Johannes Blömer
Jan Bobolz, Gennadij Liske

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Complexity Theory

SS 2016

Homework 3

Exercise 1 (10 points)**:**
Consider the following way of defining graphs implicitly. Let Π be a set of symbols and let $P : \Pi^* \times \Pi^* \to \{0,1\}$ be a predicate that can be decided by a linear space DTM.
We define the directed graph $G_{\Pi,P,k} := (V_k, E_k)$ whose nodes are $V_k := \Pi^k$ and whose edge relation is defined through P, i.e. $E_k := \{(a,b) \in V_k^2 \mid P(a,b) = 1\}$. We define the language

$$Path_{\Pi,P} := \{\langle k,u,v,t\rangle \mid u,v \in \Pi^k \wedge \text{there is a directed path from } u \text{ to } v \text{ of length at most } t \text{ in } G_{\Pi,P,k}\}.$$

a) Show that $Path_{\Pi,P}$ can be decided in $\mathcal{O}(k \cdot \log(t))$ space by giving an algorithm and proving its space requirements.
Hint: you can borrow heavily from the Canyield *procedure in the lecture.*

b) Prove Savitch's theorem: Let $s(n)$ be a space constructible function, then it holds that $\mathbf{NSPACE}(s(n)) \subseteq \mathbf{DSPACE}(s(n)^2)$.

  (a) Specify Π, P for any given $s(n)$ space NTM N.

  (b) Reduction: For $w \in \Sigma^*$, what are k,u,v,t such that $w \in L(N) \Leftrightarrow \langle k,u,v,t\rangle \in Path_{\Pi,P}$?

  (c) Specify a DTM deciding $L(N)$ in space $s(n)^2$.

  (d) Where did you use that $s(n)$ is space constructible?

Exercise 2 (10 points)**:**
Let ϕ be a 3cnf formula, i.e. a Boolean formula in conjunctive normal form where every clause consists of exactly three literals. An *imperfect assignment* for ϕ is an assignment $(x_1,\ldots,x_n) \in \{0,1\}^n$ such that $\phi(x_1,\ldots,x_n) = 1$ but in every clause there is at least one unsatisfied literal (e.g., $(x_1,x_2,x_3) = (1,0,0)$ is an imperfect assignment for $(x_1 \vee \overline{x_2} \vee x_3)$ but not for $(x_1 \vee \overline{x_2} \vee \overline{x_3})$).

a) Show that if $(x_1,\ldots,x_n) \in \{0,1\}^n$ is an imperfect assignment for ϕ, then its negation $(\overline{x_1},\ldots,\overline{x_n})$ is an imperfect assignment for ϕ as well.

b) Let $iSAT := \{\langle\phi\rangle \mid \text{there exists an imperfect assignment for } \phi\}$. Show that $3SAT \leq_P iSAT$.
Hint: try replacing each clause $c_i = (L_1 \vee L_2 \vee L_3)$ *(for its three literals* L_i*) with* $(L_1 \vee L_2 \vee z_i) \wedge (\overline{z_i} \vee L_3 \vee b)$ *for new variables* z_i *and a single new variable* b*.*

c) Use this result to argue that $iSAT$ is $\mathbf{NP}$-complete.