Chapter 5 - Hierarchy Theorems

- Show that giving a TM more space increases the class of problems that it can solve (Space Hierarchy Theorem).
- Show that giving a TM (significantly) more time increases the class of problems that it can solve (Time Hierarchy Theorem).
- Use diagonalization to prove these results.

Preliminaries

o-Notation

Let $f, g : \mathbb{N} \to \mathbb{N}$ be functions. We write $g = o(f)$ if and only if $\lim_{n\to\infty} g(n)/f(n) = 0$. Equivalently,

$$g = o(f) \Leftrightarrow \forall c > 0\ \exists n_0 \in \mathbb{N}\ \forall n \geq n_0 : f(n) \geq c \cdot g(n).$$

Definition 5.1

A function $f : \mathbb{N} \to \mathbb{N}$, where $f(n)$ is at least $\Omega\big(\log(n)\big)$, is called space constructible if the function that maps the string 1^n to the binary representation of $f(n)$ is computable in space $\mathcal{O}\big(f(n)\big)$.

Gödel numbers

Notation

For a TM M denote by $\langle M \rangle$ the Gödel number of M (in any reasonable format you like).

Theorem 5.2

The language

$$\textit{Gödel} := \{w \in \{0,1\}^* \mid w = \langle M \rangle \text{ for some TM } M\}$$

is decidable in space $\mathcal{O}\big(\log(|w|)\big)$ *and time* $\mathcal{O}\big(|w| \cdot \log(|w|)\big)$.

Universal Turing machines

Definition 5.3

A DTM U is called universal if it can simulate any Turing machine M, given the Gödel number of machine M.

Theorem 5.4

There is a universal Turing machine that can simulate a $s(n)$ space DTM M in space $c \cdot (|\langle M \rangle| + s(n))$ for some constant c.

The space hierarchy theorem

Theorem 5.5

For any space constructible function $f : \mathbb{N} \to \mathbb{N}$*, a language* A *exists that is decidable in space* $\mathcal{O}\big(f(n)\big)$ *but not in space* $o\big(f(n)\big)$*.*

Proof of the space hierarchy theorem

$D =$ "On input $w \in \{0,1\}^*$:

1. Let n be the length of w.
2. Compute $f(n)$ using space constructibility, and mark off this much tape. If later stages ever attempt to use more space, *reject*.
3. If w is not of the form $\langle M \rangle 10^*$, *reject*.
4. Simulate M on input w while counting the number of steps used in the simulation. If the count ever exceeds $2^{f(n)}$, *reject*.
5. If M accepts, *reject*. If M rejects, *accept*."

Key facts

1. D decides $L(D)$ in space $\mathcal{O}\big(f(n)\big)$.
2. $L(D)$ cannot be decided in space $o\big(f(n)\big)$.

Consequences

Corollary 5.6

For any two functions $f_1, f_2 : \mathbb{N} \to \mathbb{N}$*, where* $f_1(n)$ *is* $o\big(f_2(n)\big)$ *and* f_2 *is space constructible,*

$$\mathbf{DSPACE}\big(f_1(n)\big) \subsetneq \mathbf{DSPACE}\big(f_2(n)\big).$$

Corollary 5.7

For any two real numbers $0 < \epsilon_1 < \epsilon_2$*,*

$$\mathbf{DSPACE}(n^{\epsilon_1}) \subsetneq \mathbf{DSPACE}(n^{\epsilon_2}).$$

Corollary 5.8

$\mathbf{NL} \subsetneq \mathbf{PSPACE}$.

Time constructible functions

Definition 5.9

A function $t : \mathbb{N} \to \mathbb{N}$, where $t(n)$ is at least $\Omega\big(n \log(n)\big)$, is called time constructible if the function that maps the string 1^n to the binary representation of $t(n)$ is computable in time $\mathcal{O}\big(t(n)\big)$ (on a single tape DTM).

Remark

The condition $t(n) = \Omega(n \log(n))$ is necessary. Even a simple function like the identity requires time $\Omega(n \log(n))$ to compute on a single tape DTM.

The time hierarchy theorem

Theorem 5.10

For any time constructible function $t : \mathbb{N} \to \mathbb{N}$*, a language* A *exists that is decidable in time* $\mathcal{O}\big(t(n)\big)$ *but not in time* $o\big(t(n)/\log(t(n))\big)$.

Corollary 5.11

For any two functions $t_1, t_2 : \mathbb{N} \to \mathbb{N}$*, where* $t_1(n)$ *is* $o(t_2(n)/\log(t_2(n)))$ *and* t_2 *is time constructible,*

$$\mathbf{DTIME}(t_1(n)) \subsetneq \mathbf{DTIME}(t_2(n)).$$

Corollary 5.12

For any two numbers $1 \leq \epsilon_1 < \epsilon_2$ *we have*

$$\mathbf{DTIME}(n^{\epsilon_1}) \subsetneq \mathbf{DTIME}(n^{\epsilon_2}).$$

Between **L** and **PSPACE**

Conjecture

In the following sequence all inclusions are proper,

$$\mathbf{L} \subsetneq \mathbf{NL} \subsetneq \mathbf{P} \subsetneq \mathbf{NP} \subsetneq \mathbf{PSPACE}.$$

Necessity of constructibility

Theorem 5.13

There is a computable non-constant function $f : \mathbb{N} \to \mathbb{N}$ such that

$$\mathbf{DTIME}(f(n)) = \mathbf{DTIME}(2^{f(n)}).$$