Premaster Course Algorithms 1

Chapter 6: Shortest Paths

Christian Scheideler

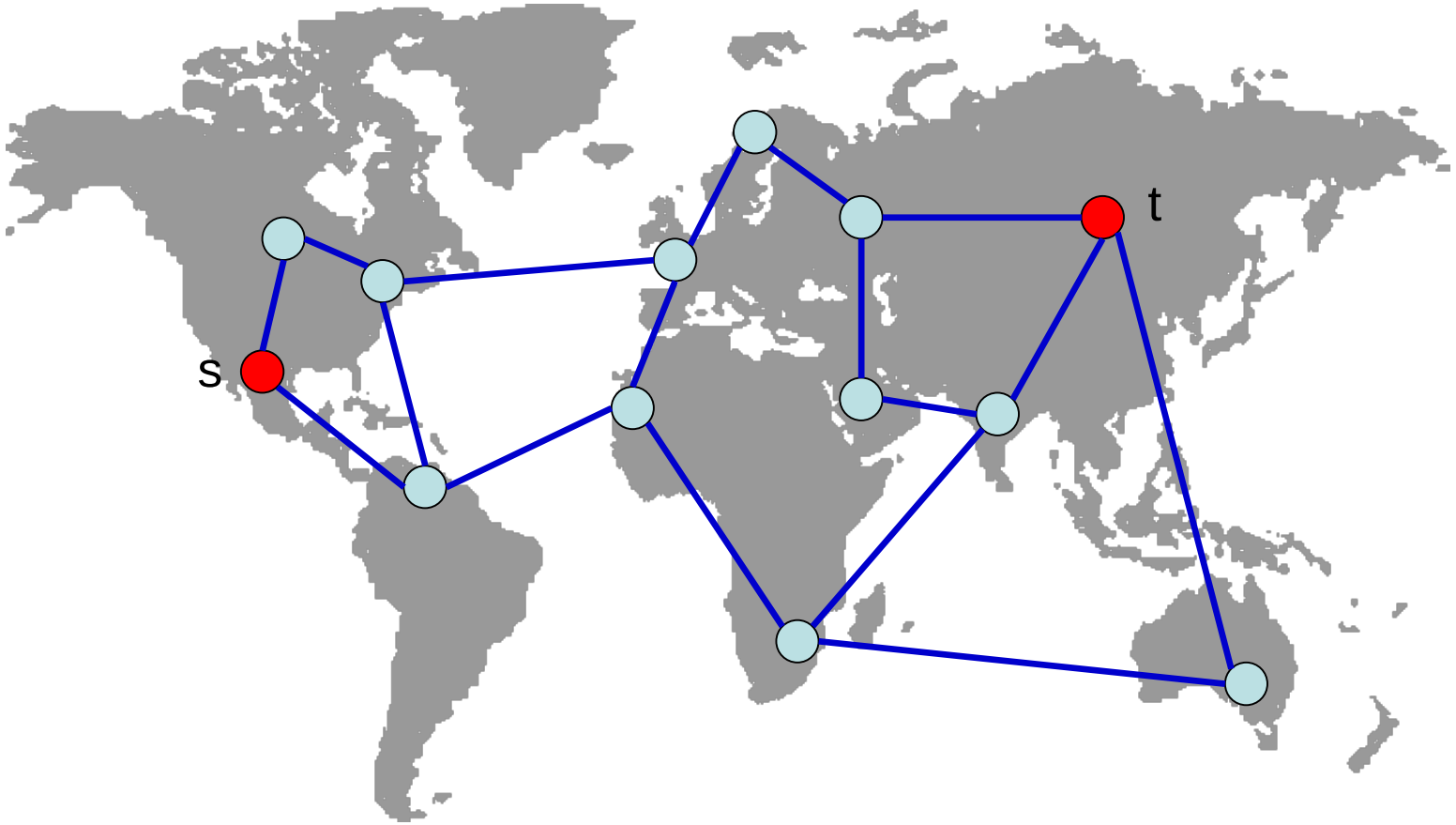
SS 2018

Basic Graph Algorithms

Overview:

- Shortest paths in DAGs
- Dijkstra‘s algorithm
- Bellman-Ford algorithm
- Johnson‘s method

Shortest Paths



Central question: Determine fastest way to get from s to t.

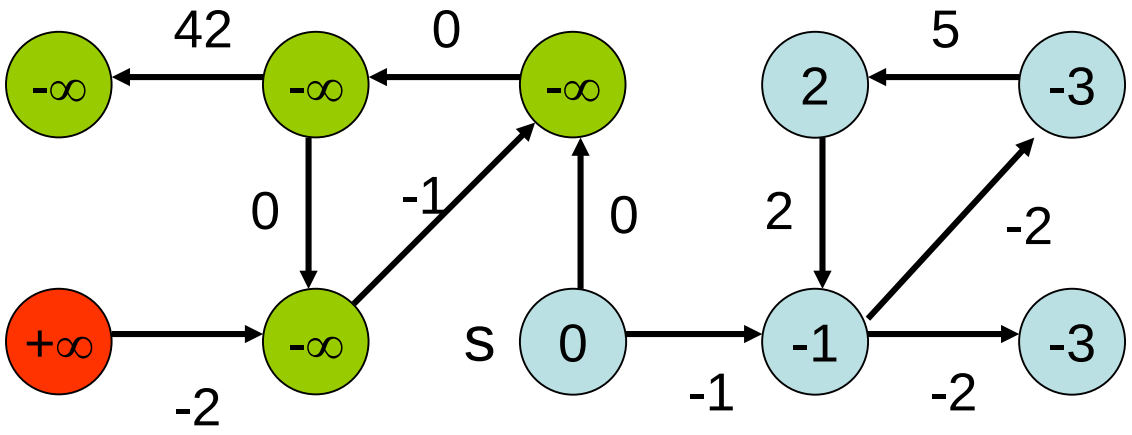
Shortest Paths

Shortest Path Problem:

- directed/undirected graph $G=(V,E)$
- edge costs $c:E \rightarrow \mathbb{R}$

- SSSP (single source shortest path): find shortest paths from a source node to all other nodes
- APSP (all pairs shortest path): find shortest paths between all pairs of nodes

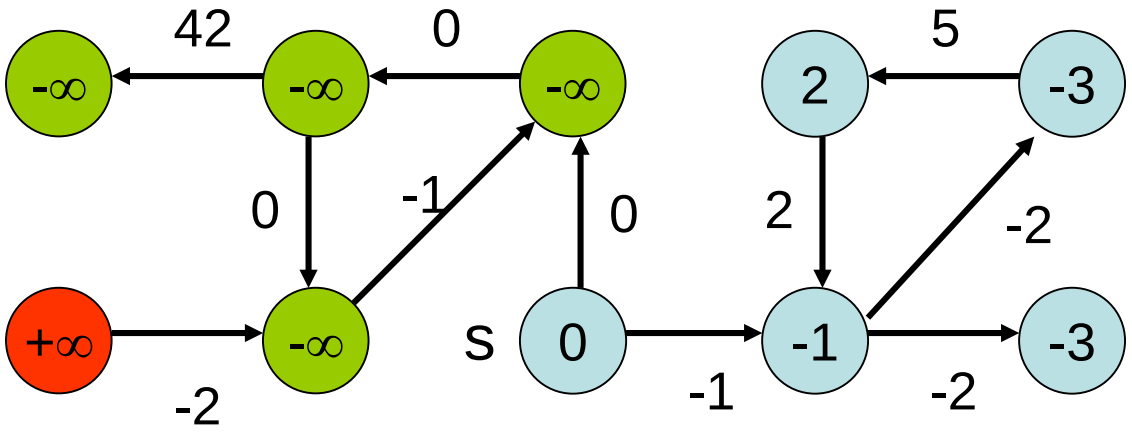
Shortest Paths



$\delta(s,v)$: distance between s and v

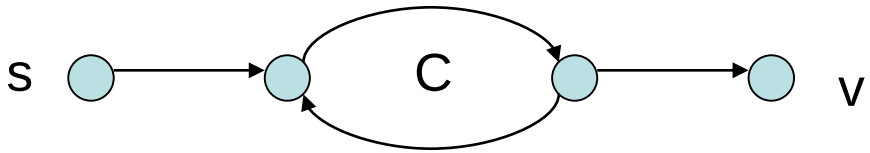
$$\delta(s,v) = \begin{cases} \infty & \text{no path from } s \text{ to } v \\ -\infty & \text{path of arbitrarily low cost from } s \text{ to } v \\ \min\{\, c(p) \mid p \text{ is a path from } s \text{ to } v\} \end{cases}$$

Shortest Paths



When is the distance -∞?

If there is a negative cycle:

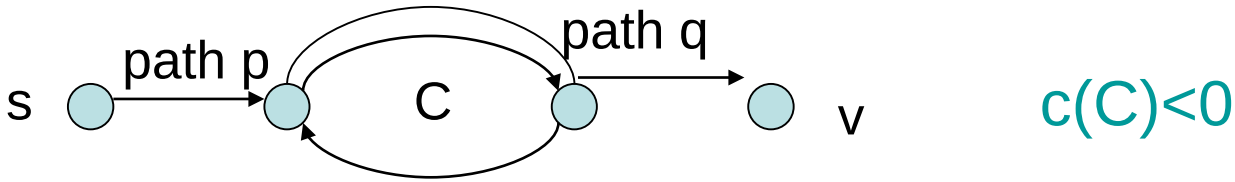


c(C)<0

Shortest Paths

Negative cycle necessary and sufficient for a distance of $-\infty$.

Negative cycle sufficient:



Cost for i-fold traversal of C:

$$c(p) + i \cdot c(C) + c(q)$$

For $i \to \infty$ this expression approaches $-\infty$.

Shortest Paths

Negative cycle necessary and sufficient for a distance of $-\infty$.

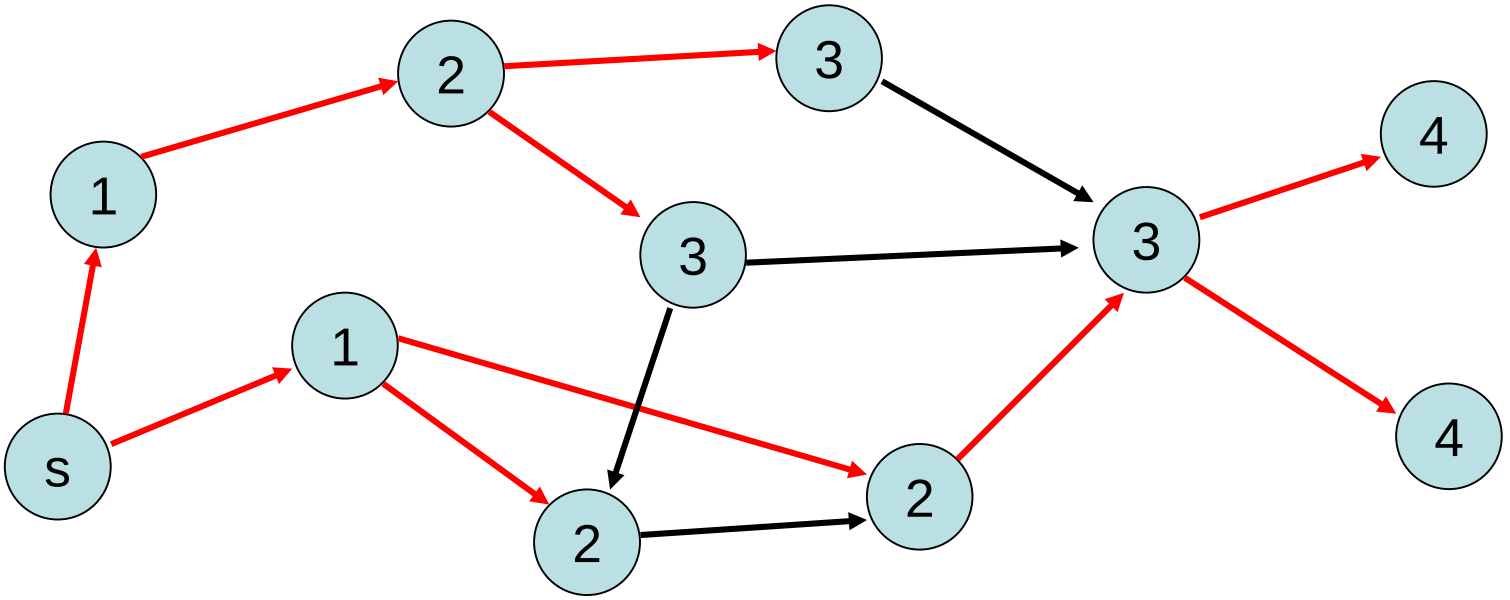
Negative cycle necessary:

- l: minimal cost of a simple path from s to v
- suppose there is a non-simple path p from s to v with cost $c(r)<l$
- p non-simple: continuously remove a cycle C till we are left with a simple path p'
- since $c(p) < l$ but $c(p') \geq l$ due to the definition of l, there must be a cycle C with $c(C)<0$

Shortest Paths in DAGs

Directed acyclic graph (DAG):

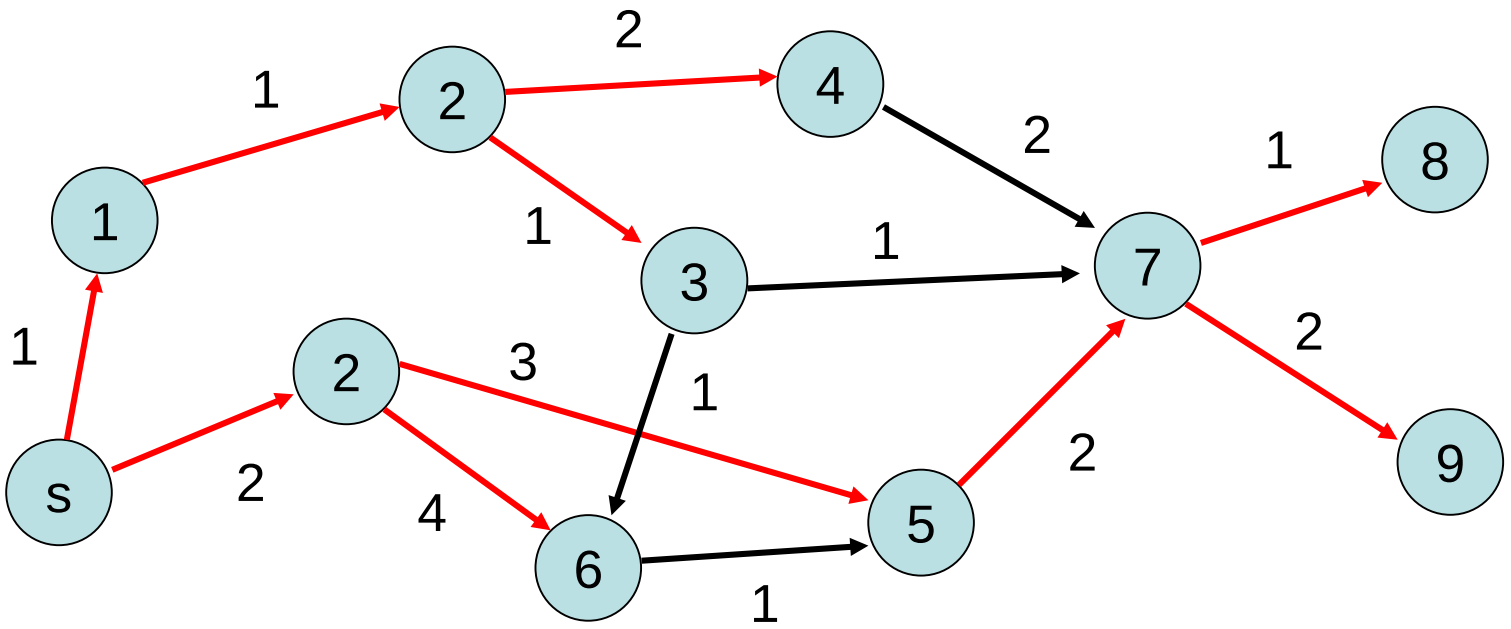
- DAG with edge costs 1: breadth-first search
  (→: edges of BFS-tree)



Shortest Paths in DAGs

Directed acyclic graph (DAG):

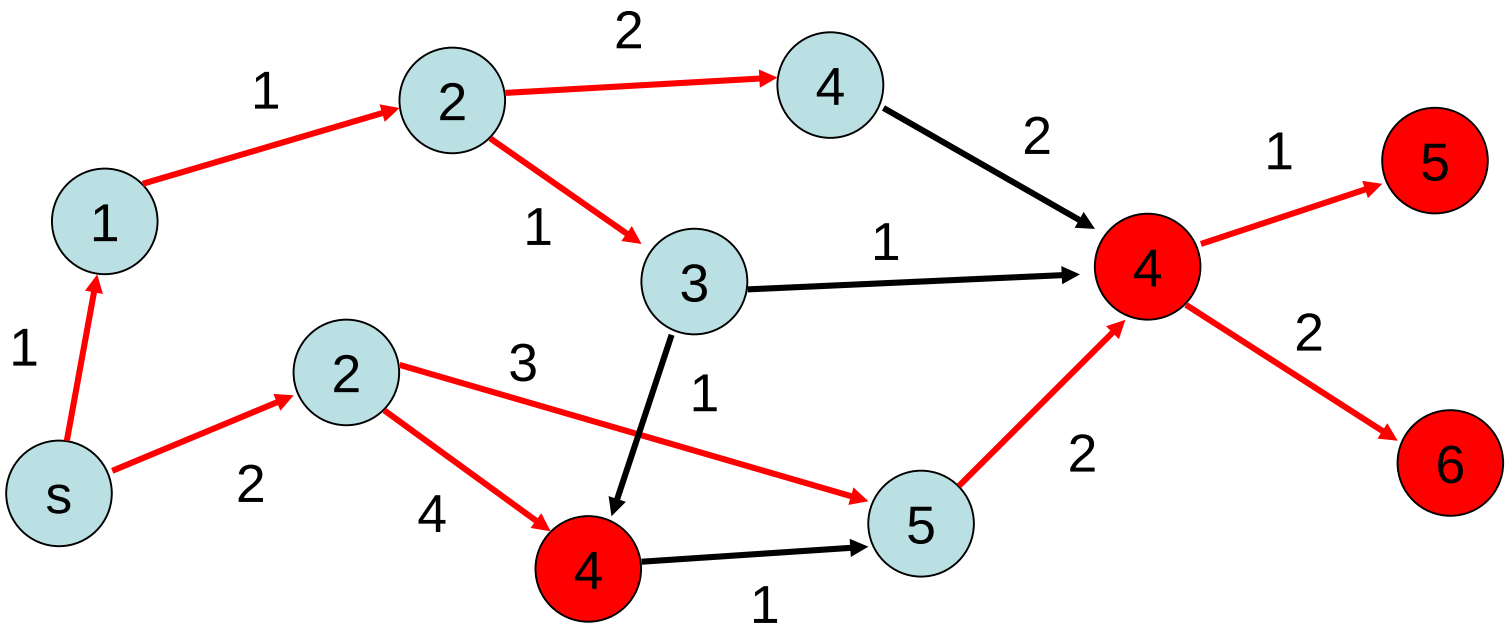
- DAG with arbitrary edge costs: breadth-first search does not work any more!



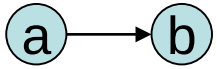
Shortest Paths in DAGs

Directed acyclic graph (DAG):

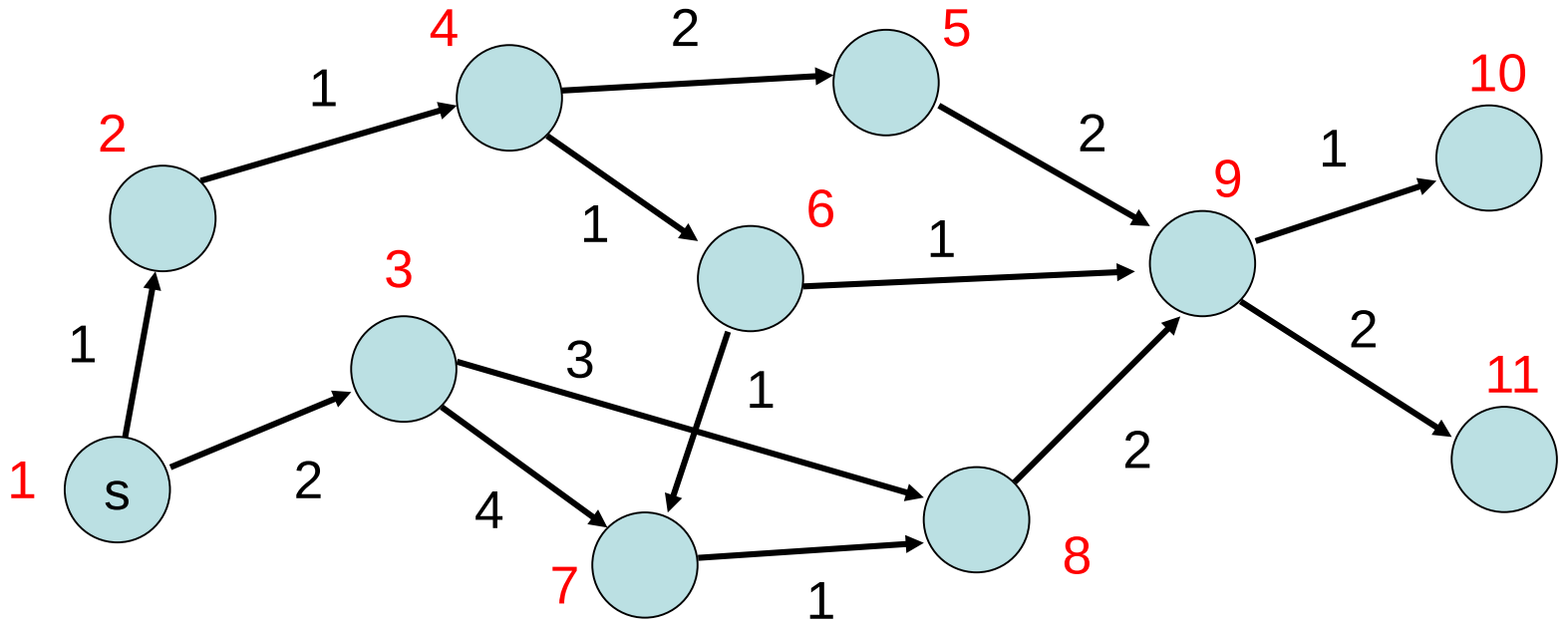
- Correct distances:



Shortest Paths in DAGs

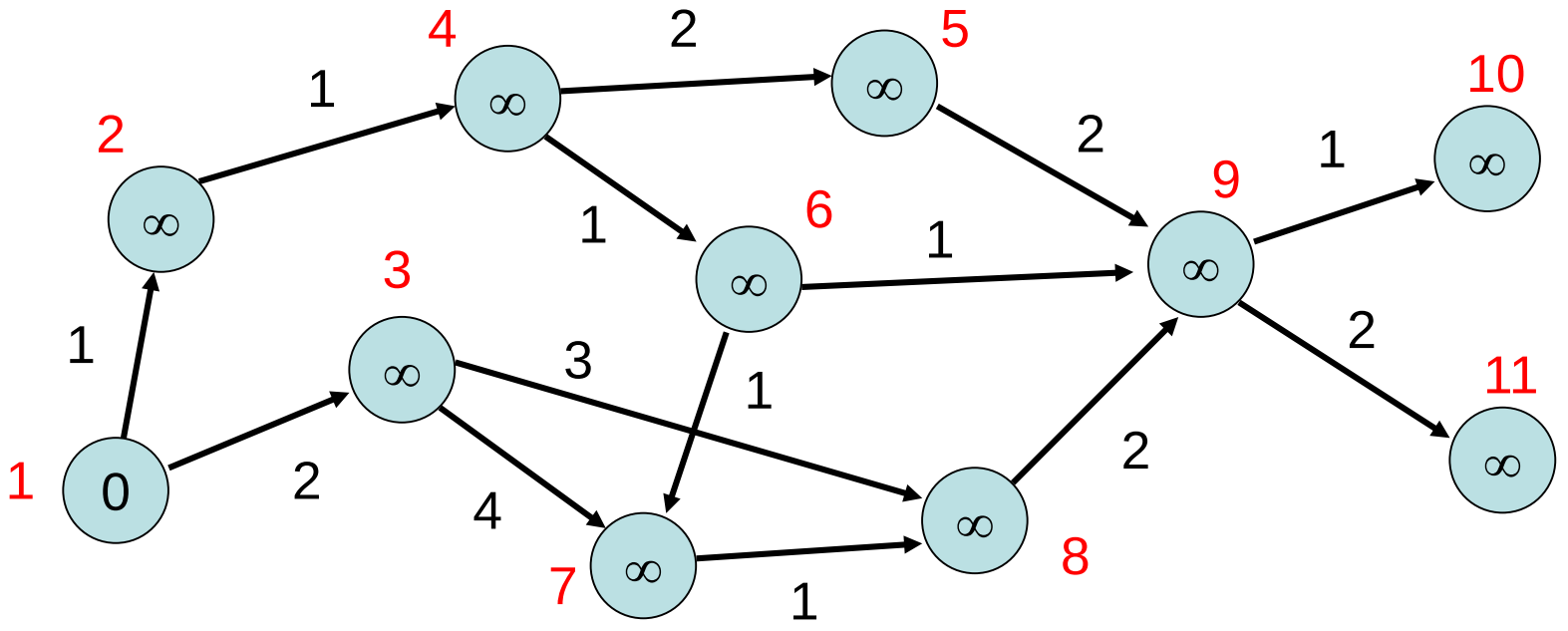
Idea: use the fact that nodes in DAGs can be topologically sorted (i.e., all edges 

satisfy a<b)



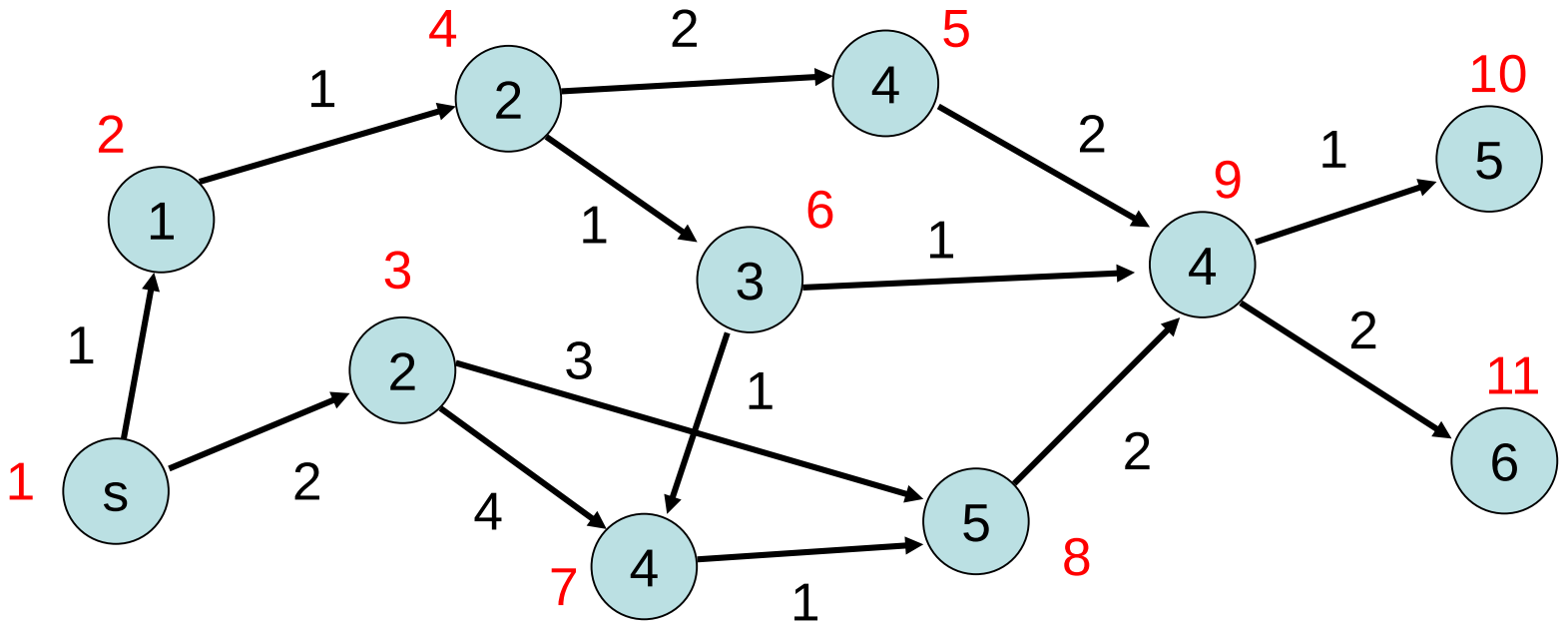
Shortest Paths in DAGs

Initially, set the distance d[s] to 0 and the distances d[v] of all other nodes v to ∞.



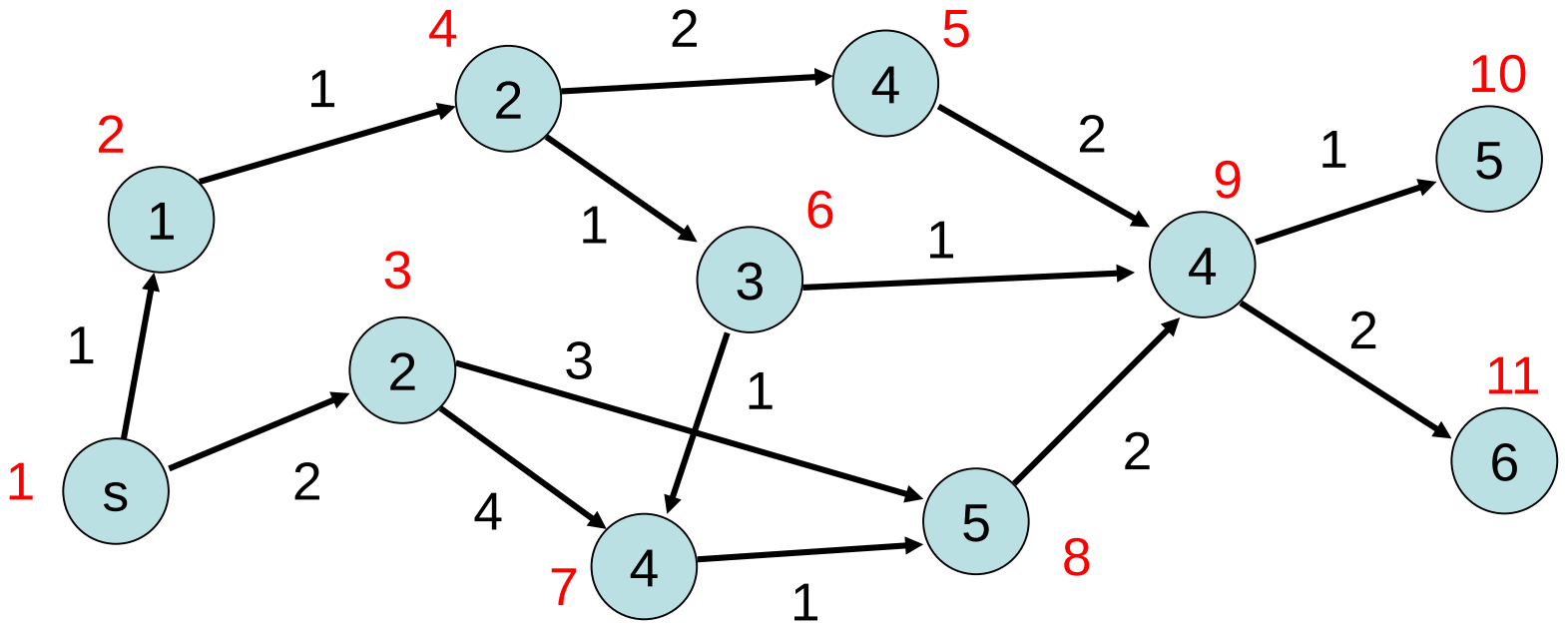
Shortest Paths in DAGs

Next, consider the nodes u in the order of the topo-logical sorting and perform distance update for u.



Shortest Paths in DAGs

Distance update for node u: for all edges $(u,v) \in E$, set d[v] to min{d[v],d[u]+c(u,v)}.



Shortest Paths in DAGs

Summary:

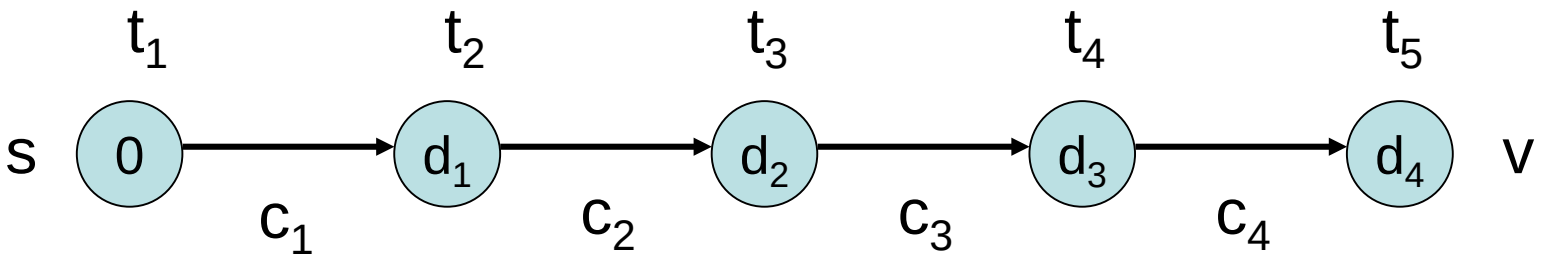
- First, compute a topological sort of the nodes (can be done via DFS).
- Then, update the distances in the order of the topological sort of the nodes.

Runtime: $O(|V|+|E|)$.

Why does that work?

Shortest Paths in DAGs

- Consider a shortest path from s to v.
- This has a topological sort with $t_i < t_{i+1}$ for all i.



- A visit in topological order therefore results in the correct distances ($d_i = \sum_{j \le i} c_i$).

Shortest Paths in DAGs

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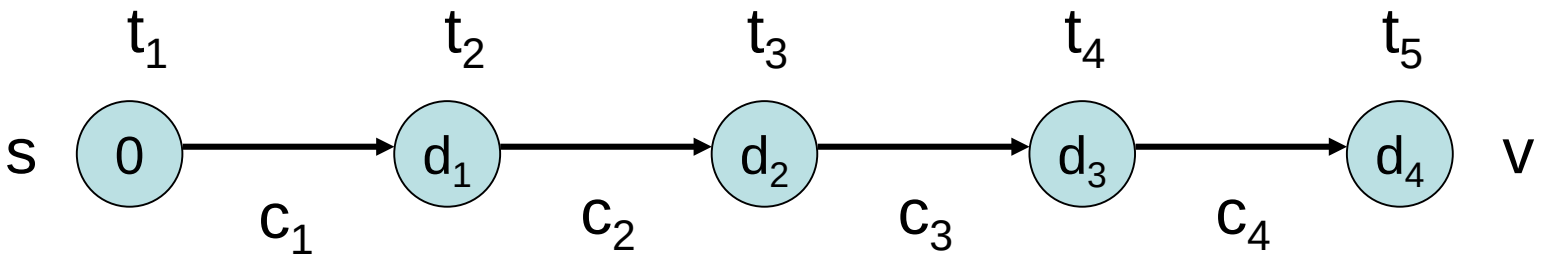
- This has a topological sort with $t_i < t_{i+1}$ for all i.



Remark: no node i along the shortest path to v can have a distance $< d_i$ to s since otherwise there is a shorter path to v.

Shortest Paths

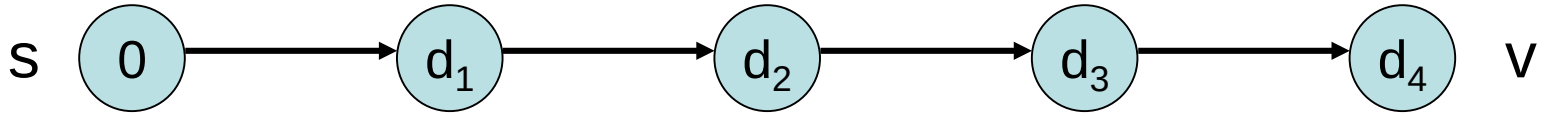
General Strategy:

- Initially, set $d(s):=0$ and $d(v):=\infty$ for all other nodes
- Visit nodes in an order that ensures that at least one shortest path from s to every v is visited in the order of its nodes
- For every visited v, update distances to nodes w with $(v,w) \in E$, i.e., $d(w):= \min\{d(w), d(v)+c(v,w)\}$

Shortest Paths

Next step: compute shortest paths for arbitrary graphs with positive edge costs.

Problem: visiting nodes of a shortest path in the right order



Solution: visit nodes in the order of their distance to s.

Shortest Paths

Assumption: all edge costs are positive. (Actually, Dijkstra also works if edge costs are non-negative.)

Dijkstra's algorithm:

1. Let S be the set of already processed nodes.
2. Initially, $S:=\{s\}$ and $d[s]:=0$
3. while $V \neq S$ do
4. choose node $v \in V \setminus S$ with at least one edge from S and $d[v]:= \min_{u \in S:(u,v) \in E} d[u] + c(u,v)$ being minimal among all $v \in V \setminus S$
5. add v to S

Shortest Paths

Assumption: all edge weights are positive.

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How to do that efficiently??

Shortest Paths

Assumption: all edge weights are positive.

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5. add v to S

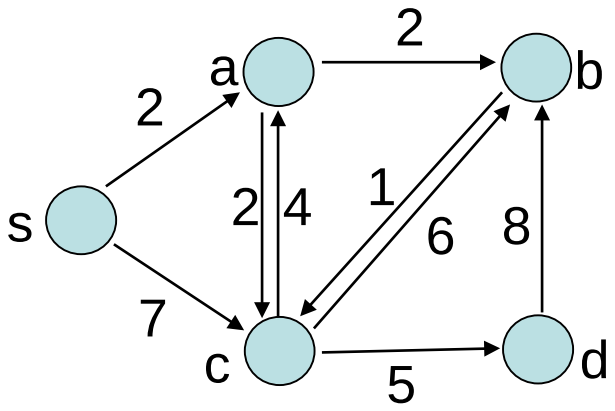
Use heap for that!

Shortest Paths

Here, edge cost of e: w(e)

Dijkstra(G,w,s)

1. **for each** vertex $v \in V$ **do** $d[v] \leftarrow \infty$; $\pi[v] \leftarrow$ **nil**
2. $d[s] \leftarrow 0$; $S \leftarrow \emptyset$; $Q \leftarrow$ BuildHeap(V)
3. **while** $Q \neq \emptyset$ **do**
4. u←deleteMin(Q)
5. $S \leftarrow S \cup \{u\}$
6. **for each** vertex $v \in$ Adj[u] **do**
7. **if** d[v]>d[u]+w(u,v) **then** DK(Q,v,d[u]+w(u,v)); $\pi[v] \leftarrow$ u



DK(Q,v,d): DecreaseKey operation
Reduces value d[v] of v in Q to d and performs heapifyUp to repair heap property from position of v.

Shortest Paths

Dijkstra(G,w,s)

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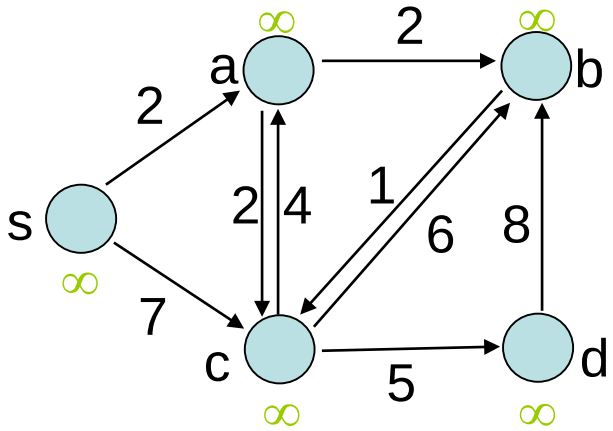

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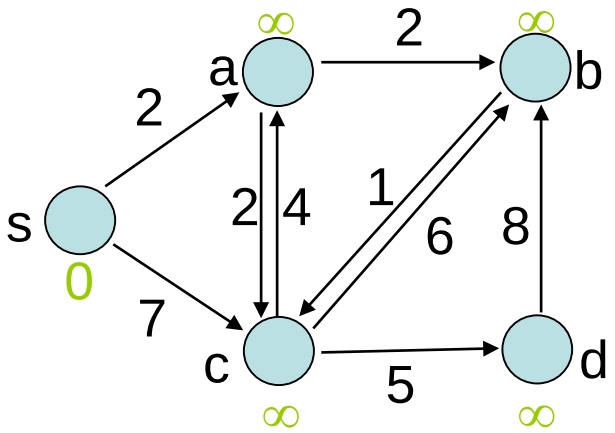

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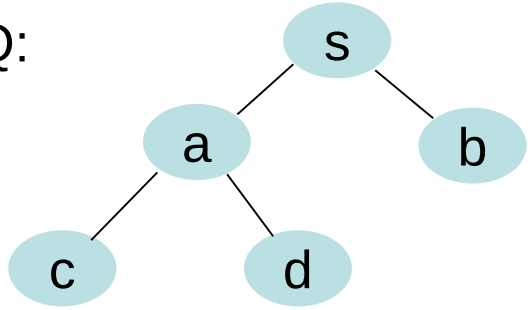

Shortest Paths

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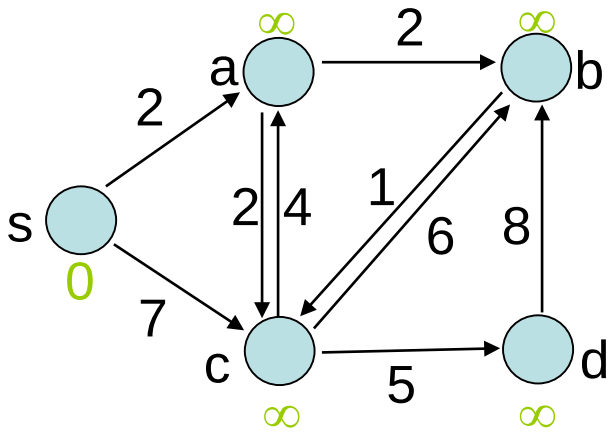

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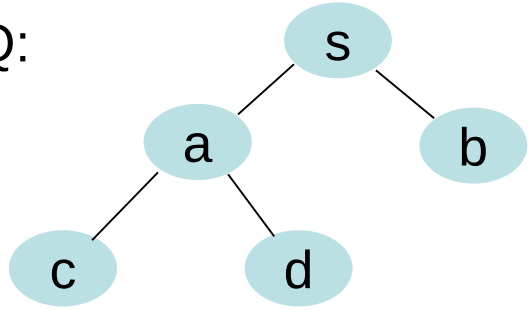

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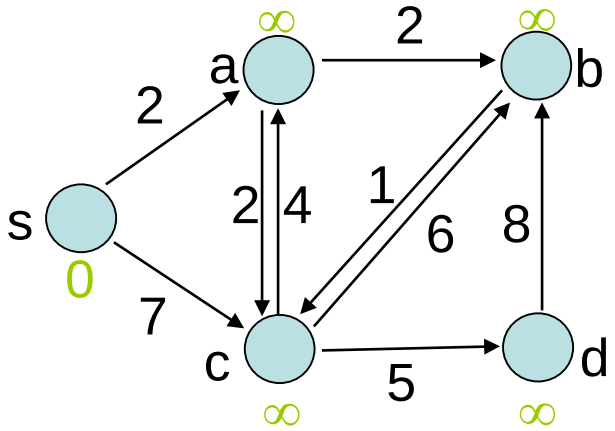

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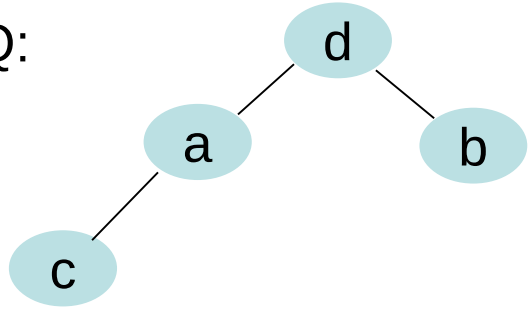

Shortest Paths

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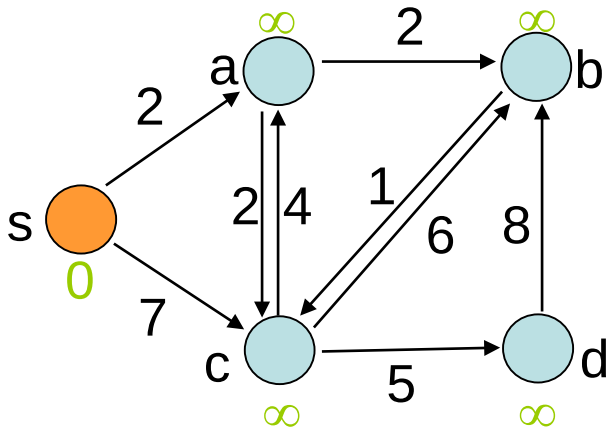

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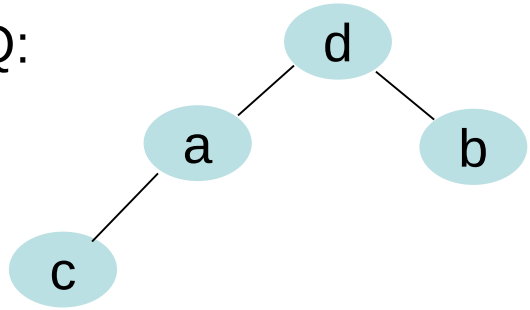

Shortest Paths

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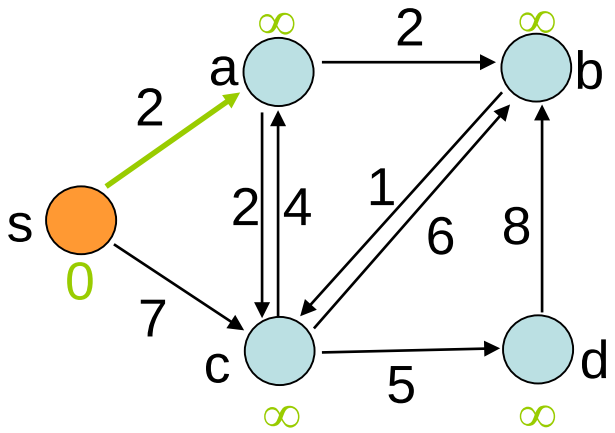

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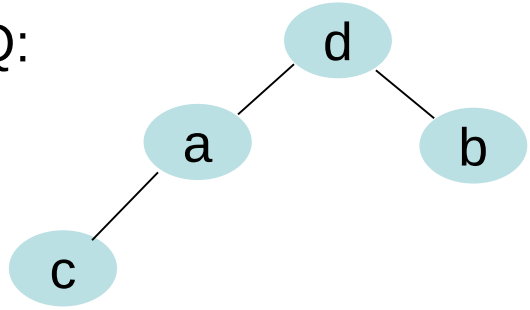

Shortest Paths

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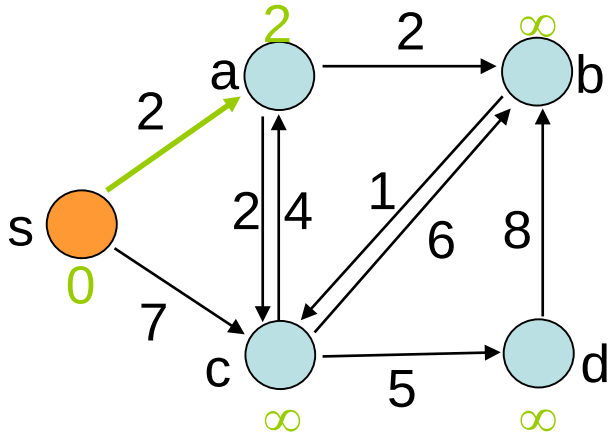

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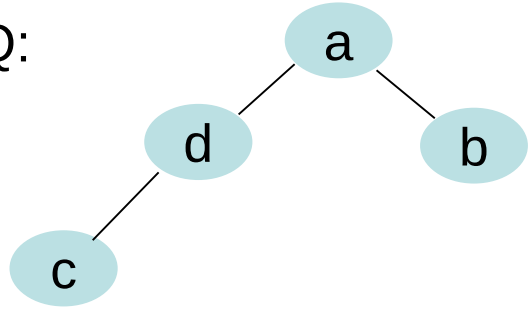

Shortest Paths

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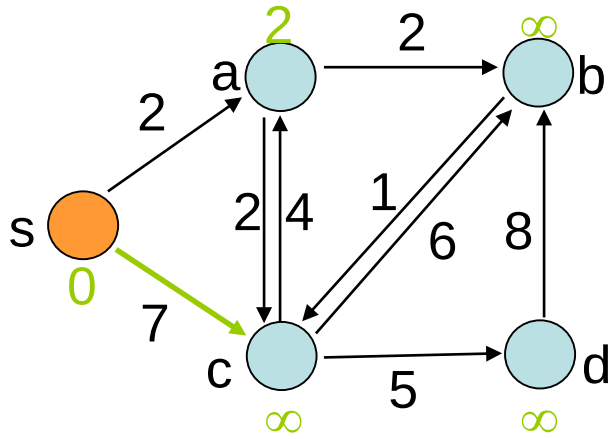

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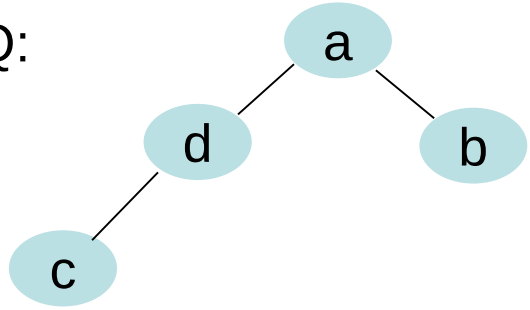

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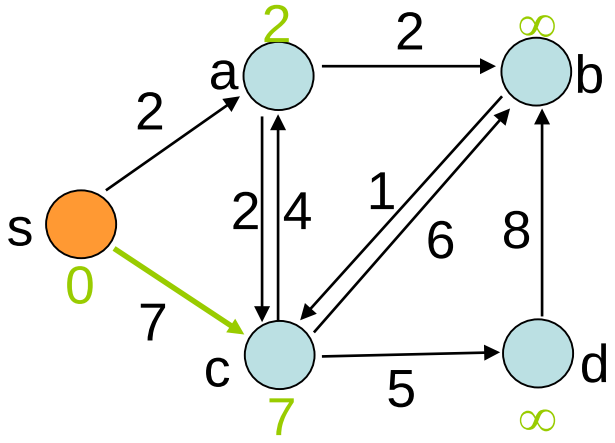
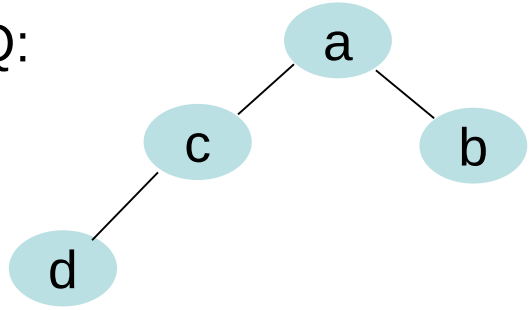


Q:



Shortest Paths

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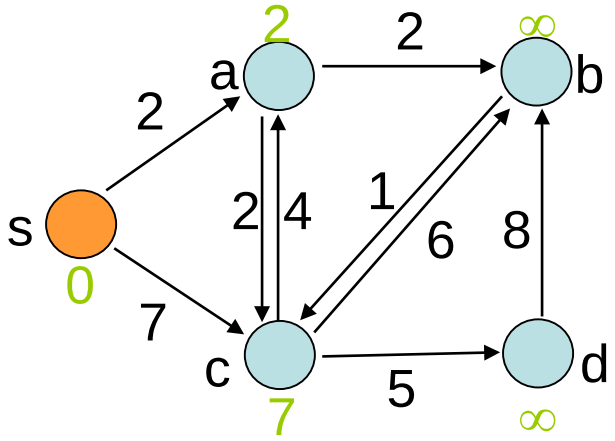

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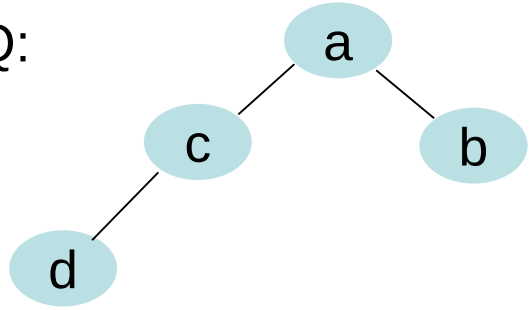

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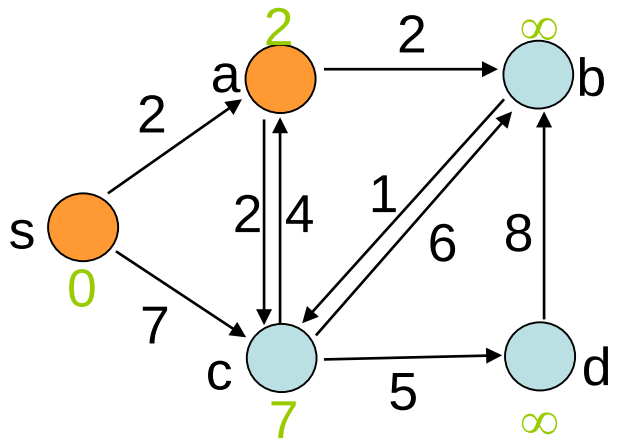

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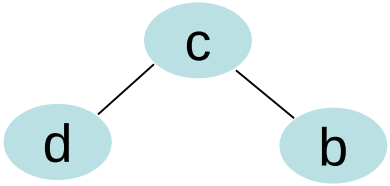

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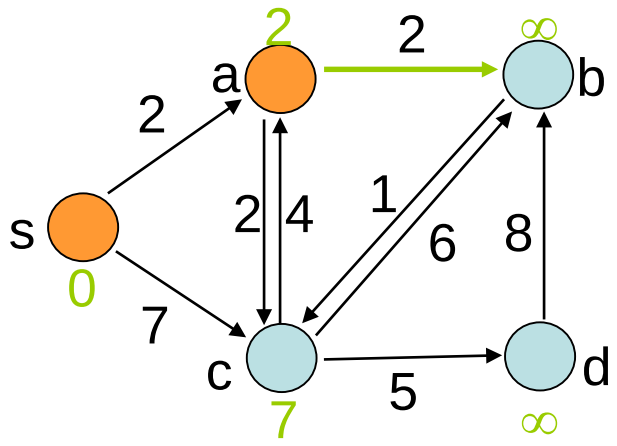

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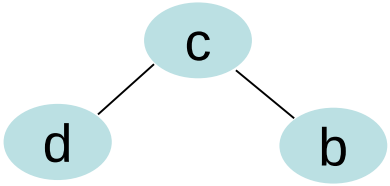

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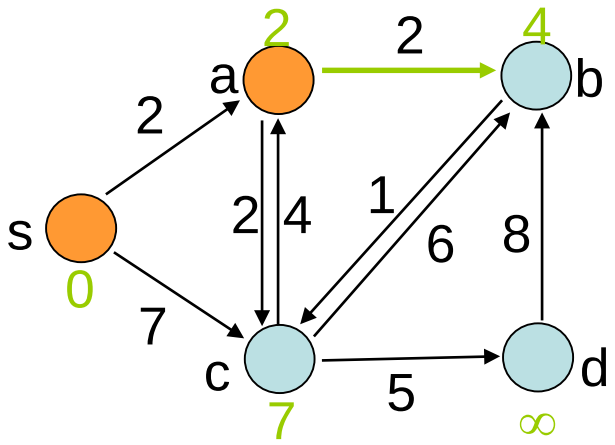

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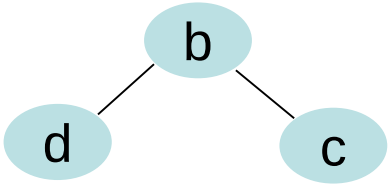

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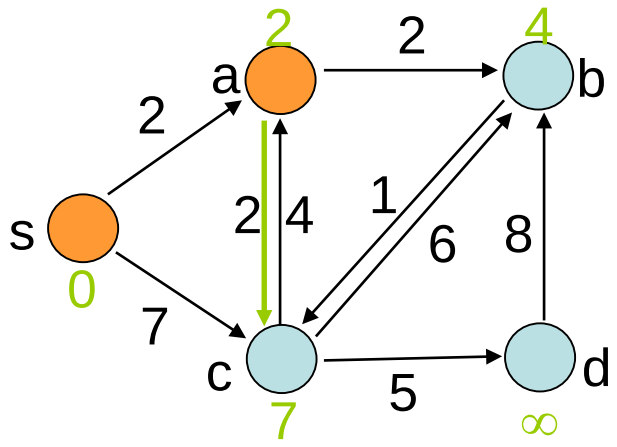

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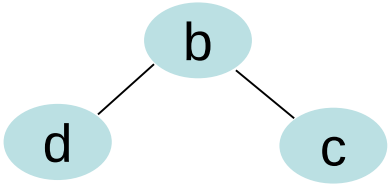

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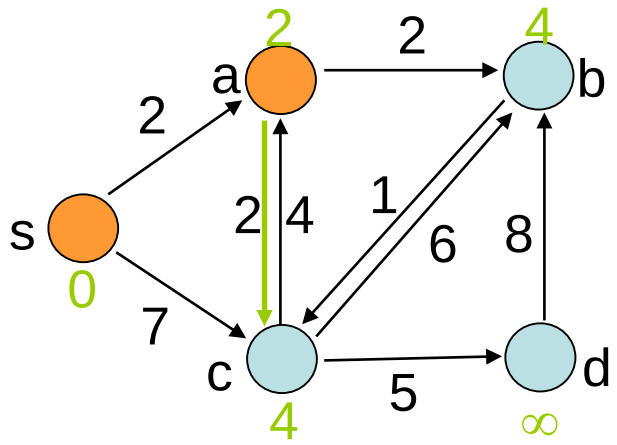

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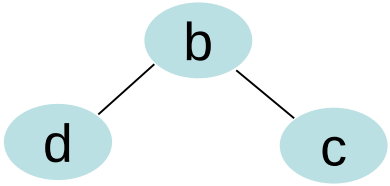

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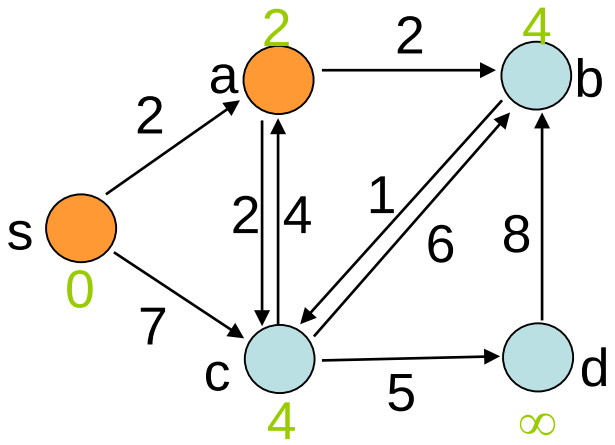

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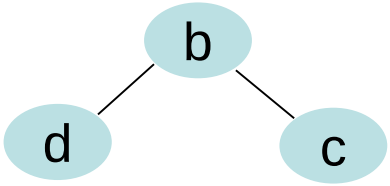

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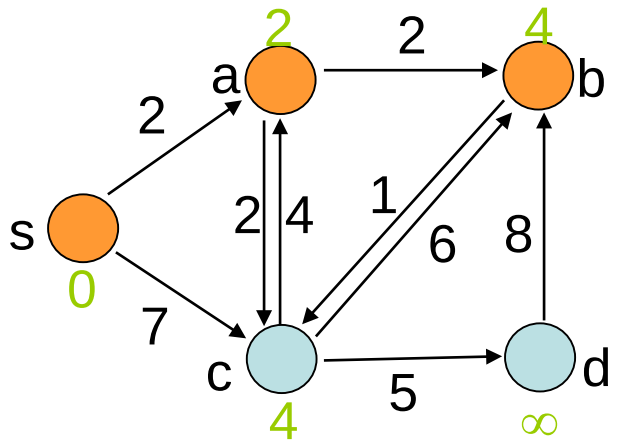

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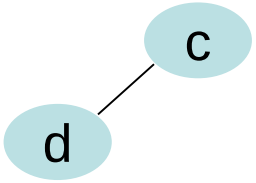

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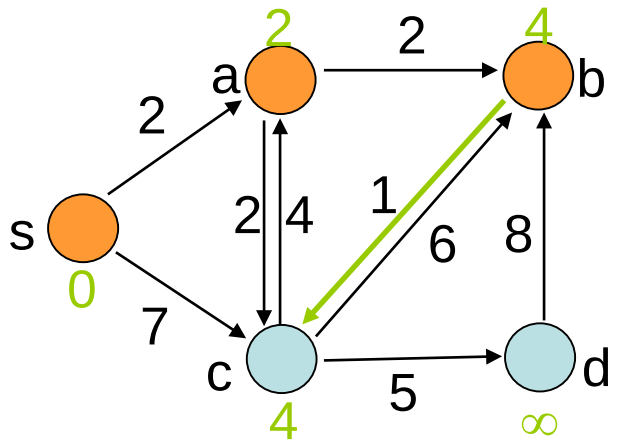

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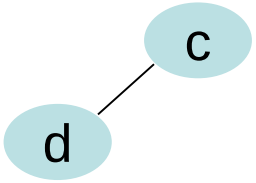

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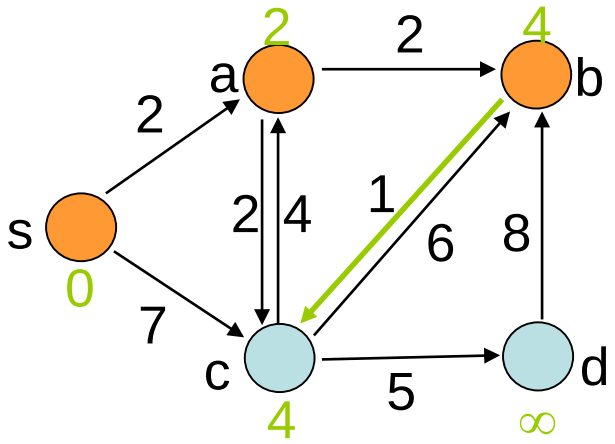

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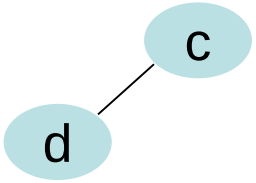

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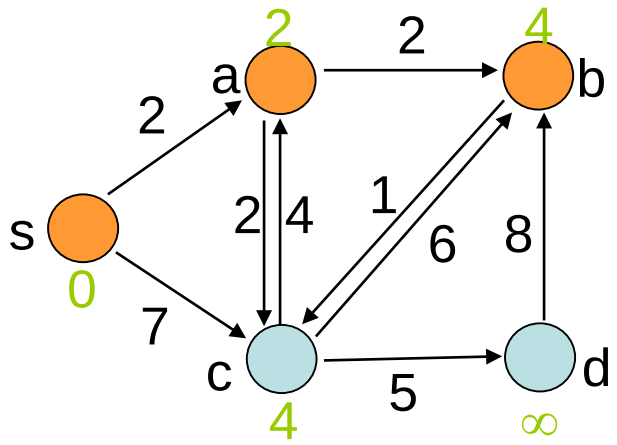

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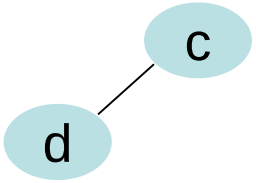

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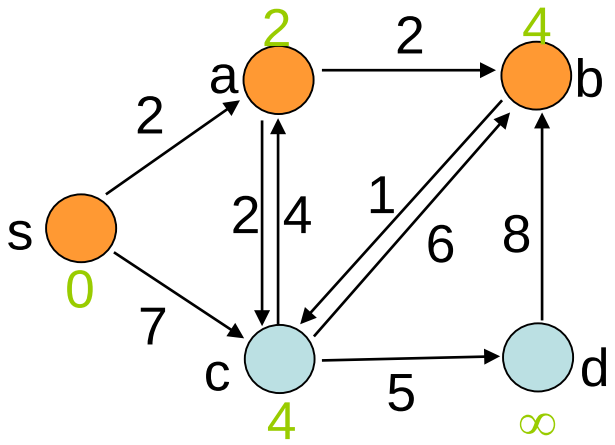

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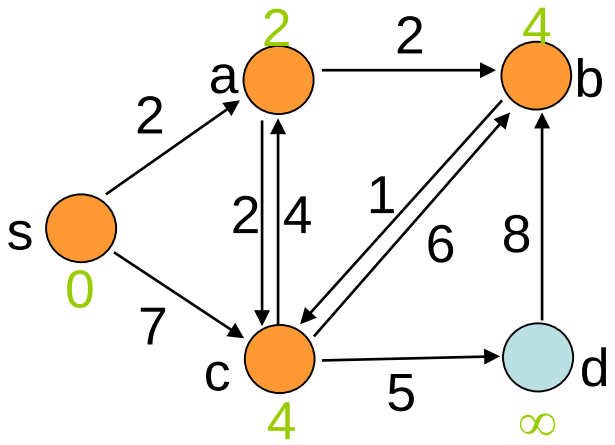

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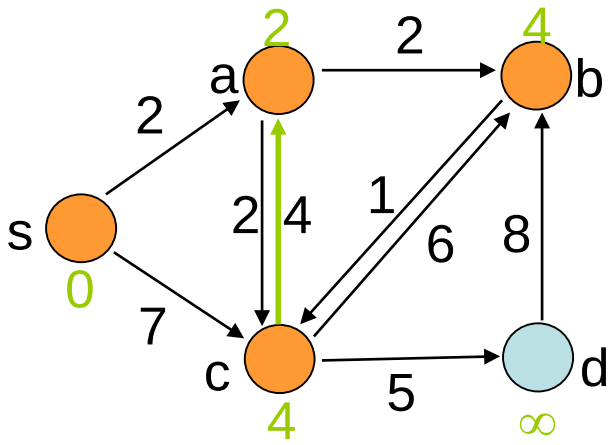

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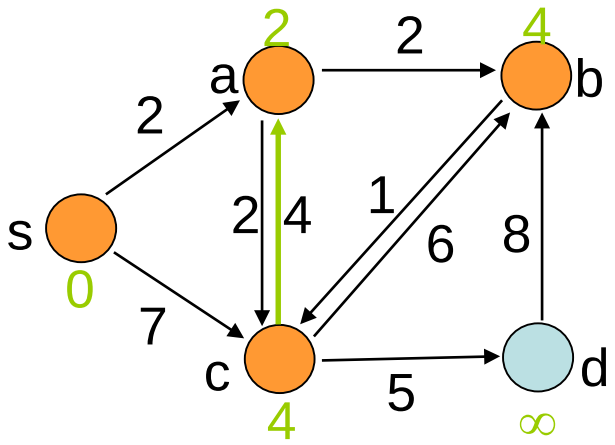

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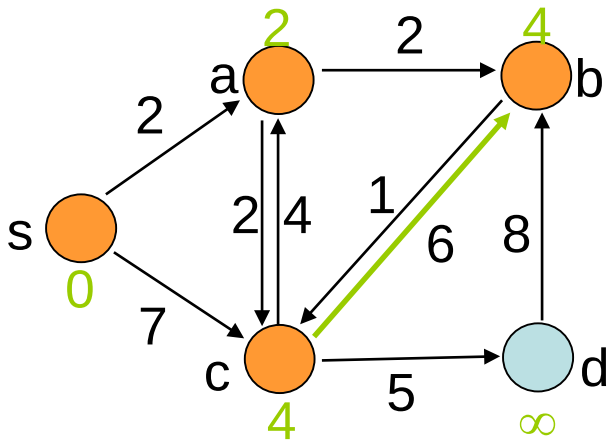

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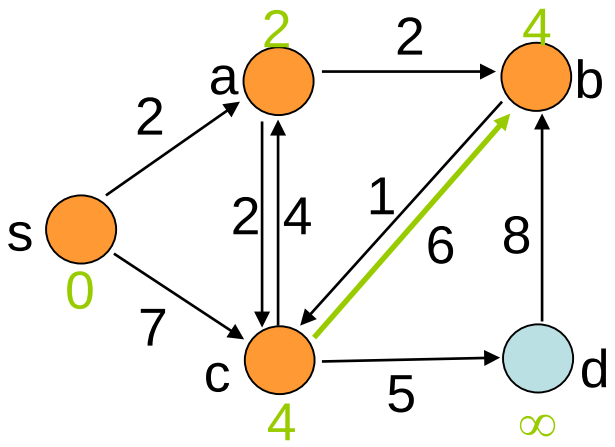

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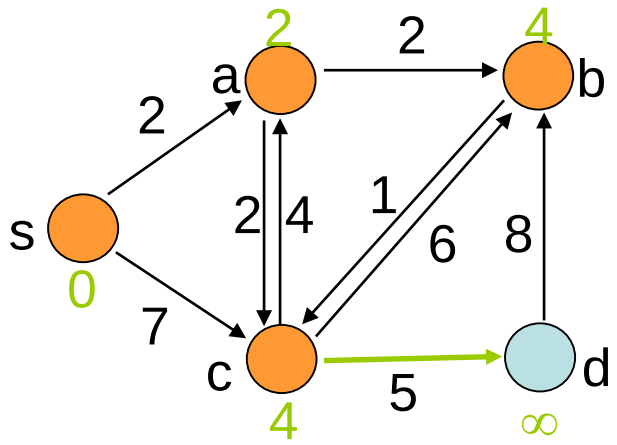

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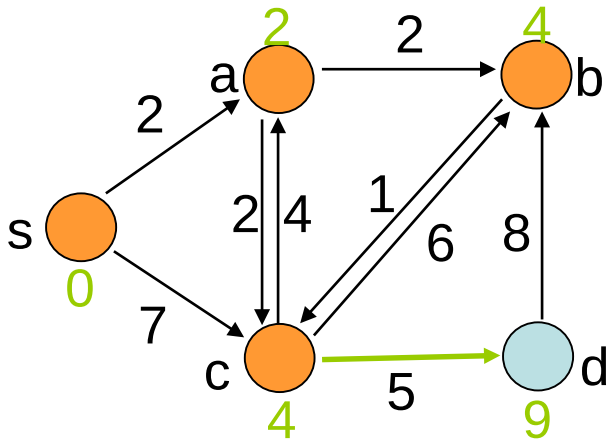

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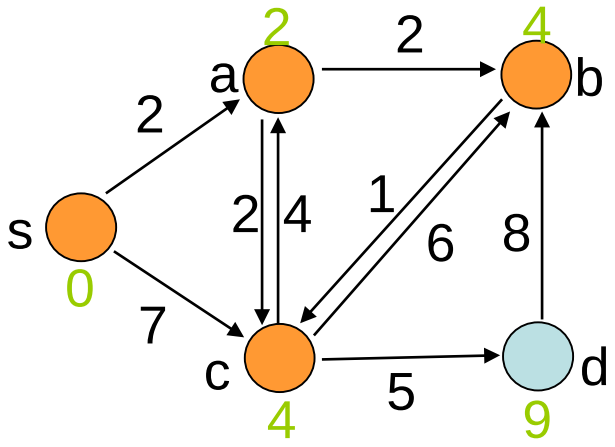

Q: d

Shortest Paths

Dijkstra(G,w,s)

1. **for each** vertex $v \in V$ **do** $d[v] \leftarrow \infty$; $\pi[v] \leftarrow$ **nil**
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3. **while** $Q \neq \emptyset$ **do**
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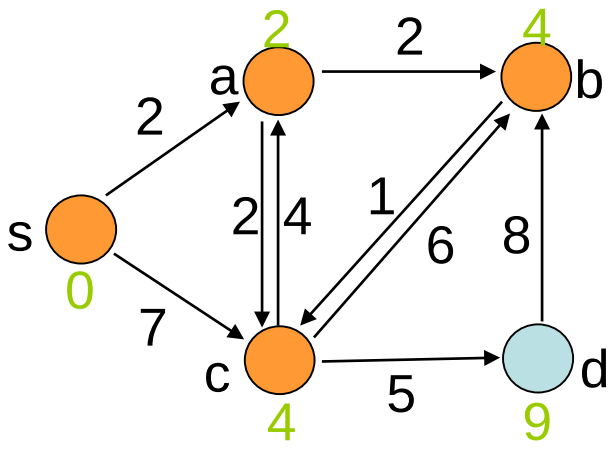

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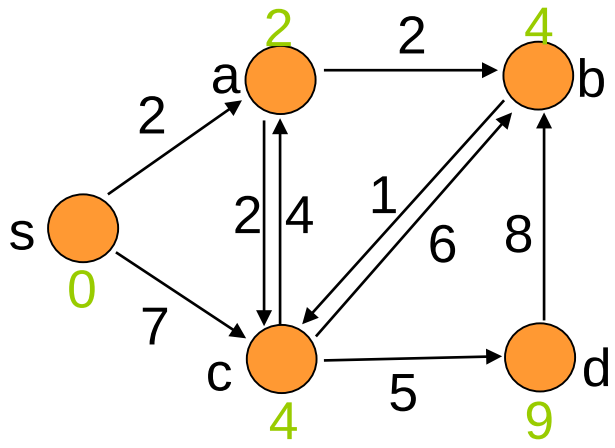

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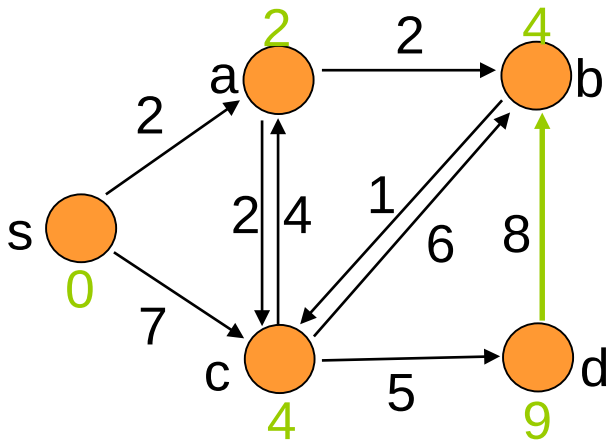

Q:

Shortest Paths

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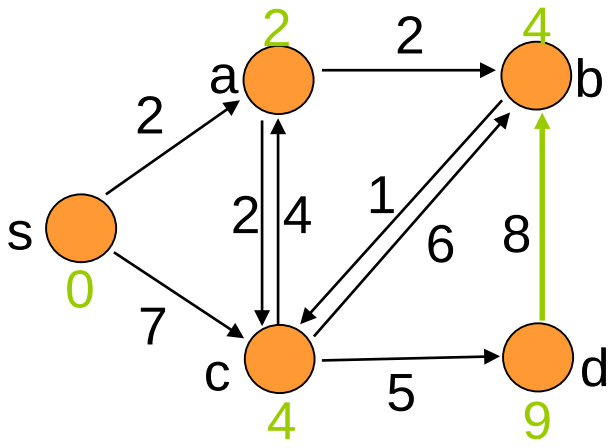

Q:

Shortest Paths

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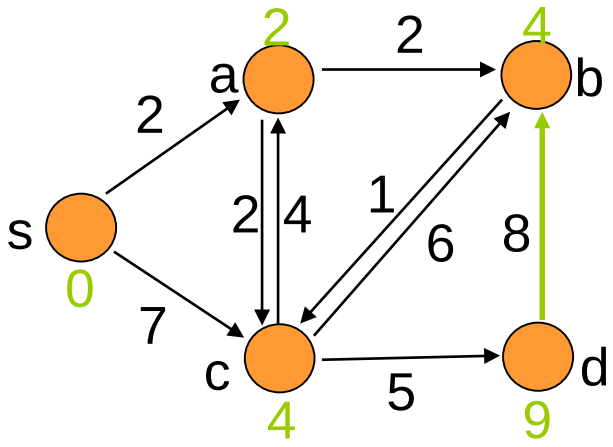

Runtime:

- Q← BuildHeap(V) (sets up heap): O(|V|)

Shortest Paths

Dijkstra(G,w,s)

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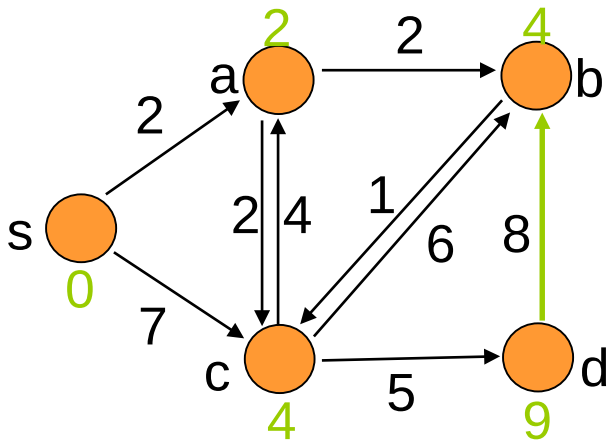

Runtime:

- deleteMin und DecreaseKey: O(log |V|)

Shortest Paths

Dijkstra(G,w,s)

1. **for each** vertex $v \in V$ **do** $d[v] \leftarrow \infty$; $\pi[v] \leftarrow$ **nil**
2. $d[s] \leftarrow 0$; $S \leftarrow \emptyset$; $Q \leftarrow V$
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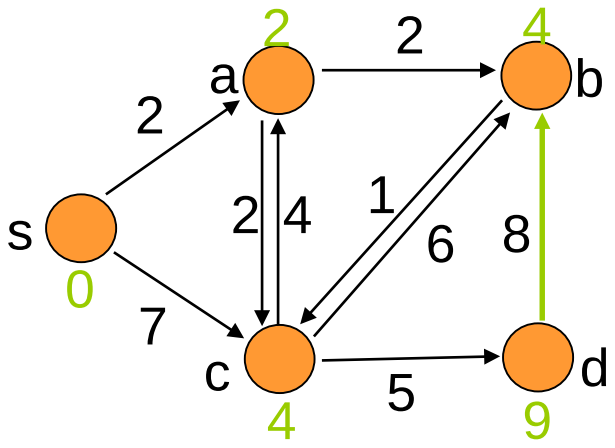

Runtime:

- deleteMin: |V| times, DecreaseKey: |E| times

Shortest Paths

Dijkstra(G,w,s)

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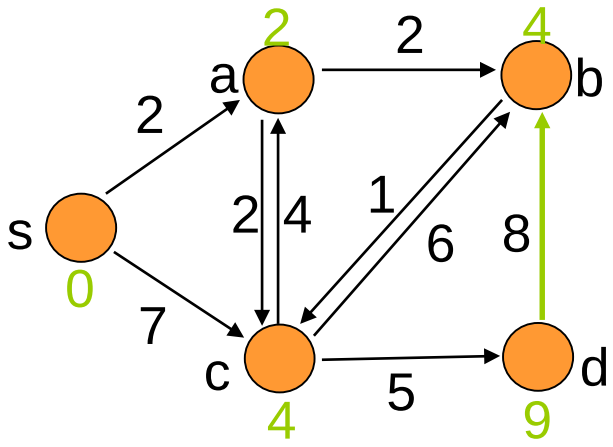


Runtime:

- altogether, $O((|V|+|E|) \log |V|)$

Shortest Paths in Arbitrary Graphs

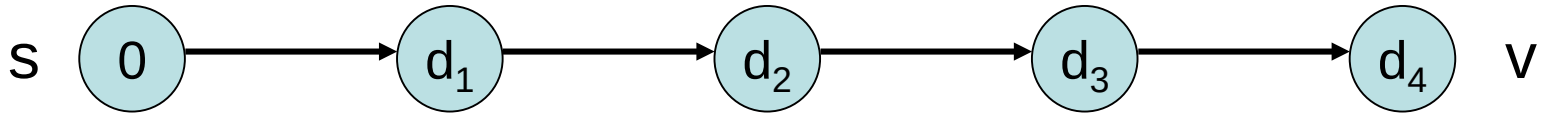
General Strategy:

- Initially, set $d(s):=0$ and $d(v):=\infty$ for all other nodes
- Visit nodes in an order that ensures that at least one shortest path from s to every v is visited in the order of its nodes
- For every visited v, update distances to nodes w with $(v,w)\in E$, i.e., $d(w):= \min\{d(w), d(v)+c(v,w)\}$

Bellman-Ford Algorithm

Consider graphs with arbitrary edge costs.

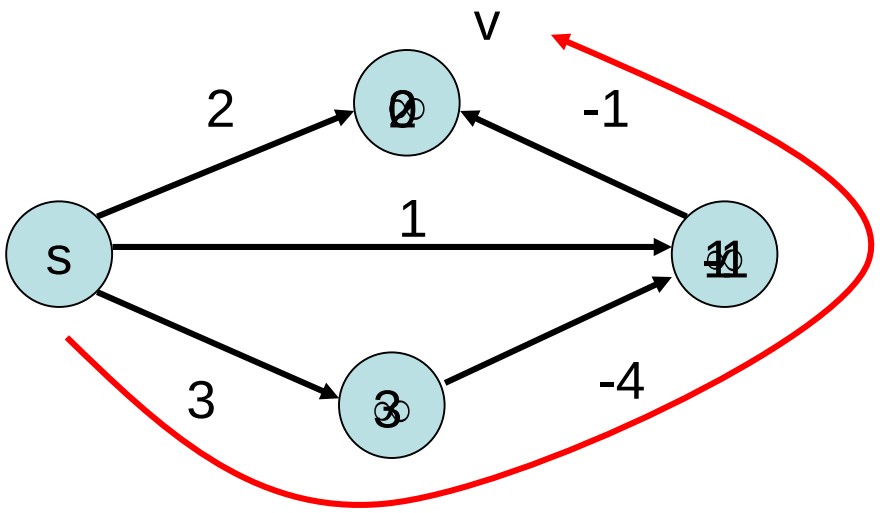
Problem: visit nodes along a shortest path from s to v in the right order



Dijkstra´s algorithm cannot be used in this case any more.

Bellman-Ford Algorithm

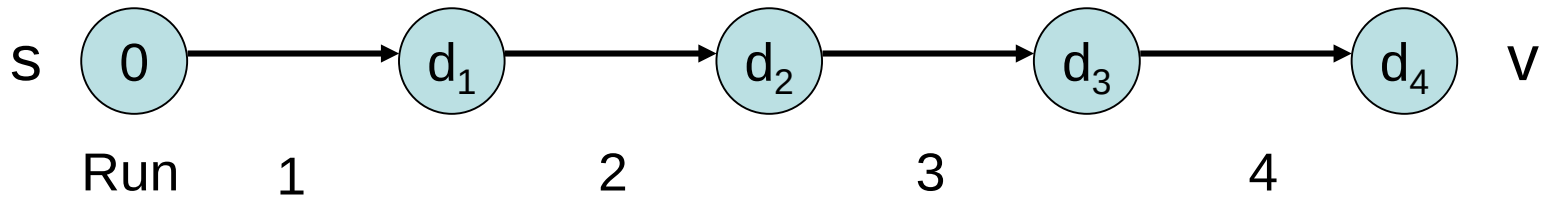
Example:



Node v has wrong distance value!

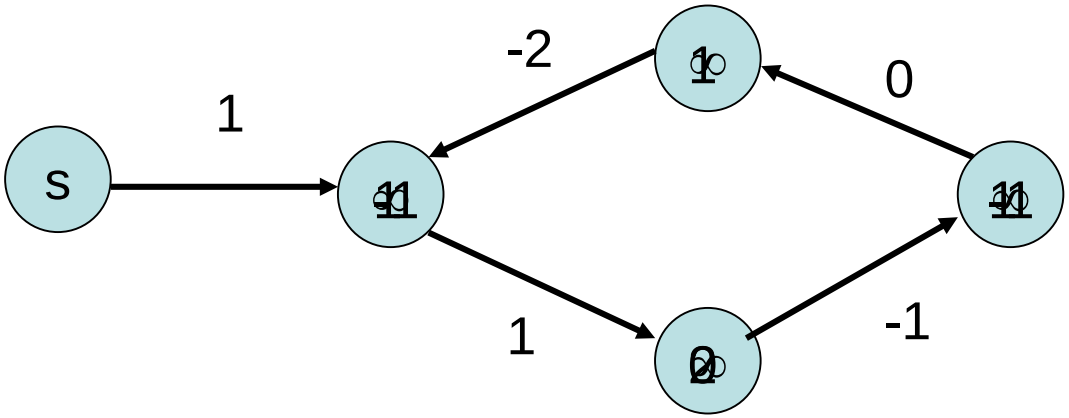
Bellman-Ford Algorithm

Strategy: visit (n-1)-times all nodes in the graph and update distances. Then all shortest paths have been considered.



Bellman-Ford Algorithm

Problem: detection of negative cycles



Conclusion: in a negative cycle, distance of at least one node keeps decreasing in each round, starting with a round <n

Bellman-Ford Algorithm

Lemma 1:

- No decrease of a distance in a round (i.e., $d[v]+c(v,w) \geq d[w]$ for all w):
  Done because $d[w]=\delta(s,w)$ for all w
- Decrease of a distance even in n-th round (i.e., $d[v]+c(v,w)<d[w]$ for some w):
  There are negative cycles for all of these nodes, so node w has distance $\delta(s,w)=-\infty$. If this is true for w, then also for all nodes reachable from w.

Proof: exercise

Bellman-Ford Algorithm

```
Procedure BellmanFord(s: NodeId)
    d:=<∞,…,∞>: NodeArray of ℝ∪{-∞, ∞}
    parent:=<⊥,…, ⊥>: NodeArray of NodeId
    d[s]:=0; parent[s]:=s
    for i:=1 to n-1 do  // update distances for n-1 rounds
        forall e=(v,w)∈E do
            if d[w] > d[v]+c(e) then  // better distance?
                d[w]:=d[v]+c(e); parent[w]:=v
    forall e=(v,w) ∈E do        // still better in n-th round?
        if d[w] > d[v]+c(e) then infect(w)

Procedure infect(v)  // set -∞-distance starting with v
    if d[v]>-∞ then
        d[v]:=-∞
        forall (v,w)∈E do infect(w)
```

Bellman-Ford Algorithm

Runtime: $O(n \cdot m)$

Improvements:

- Check in each update round if we still have $d[v]+c[v,w]<d[w]$ for some $(v,w) \in E$. No: done!
- Visit in each round only those nodes w with some edge $(v,w) \in E$ where $d[v]$ has decreased in the previous round.

All Pairs Shortest Paths

Assumption: graph with arbitrary edge costs, but no negative cycles

Naive Strategy for a graph with n nodes: run n times Bellman-Ford Algorithm (once for every node as the source)

Runtime: $O(n^2 m)$

All Pairs Shortest Paths

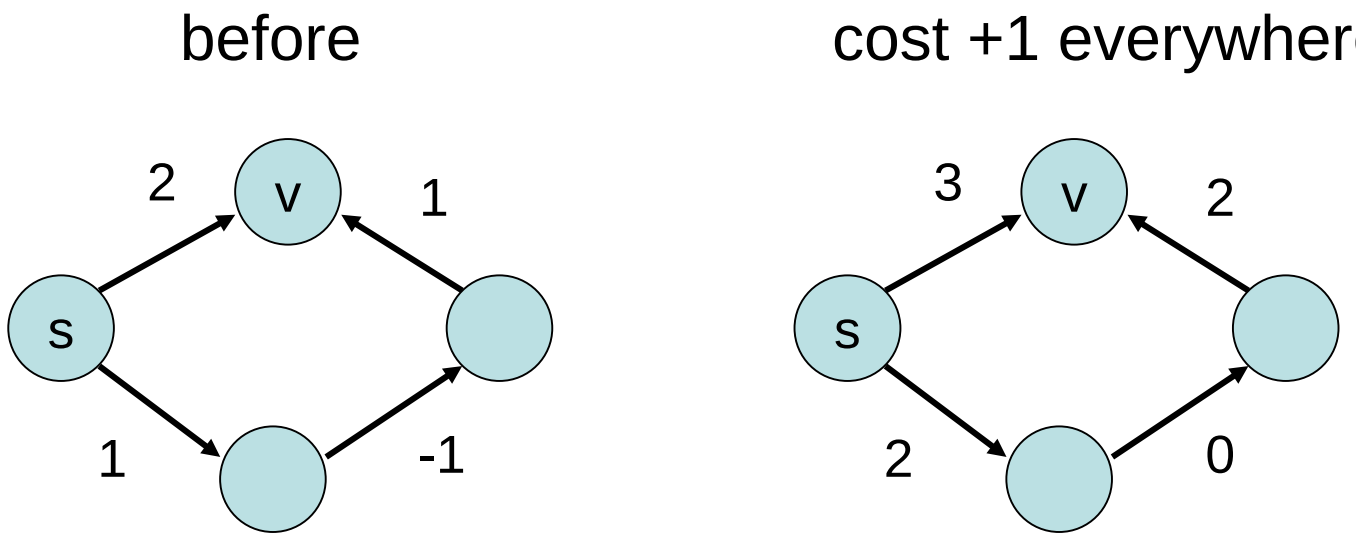
Better Strategy: Reduce n Bellman-Ford applications to n Dijkstra applications

Problem: we need non-negative edge costs

Solution: convert edge costs into non-negative edge costs without changing the shortest paths (not so easy!)

All Pairs Shortest Paths

Counter example to additive increase by c:



Johnson´s Method

- Let $\phi: V \rightarrow \mathbb{R}$ be a function that assigns a potential to every node.
- The reduced cost of $e=(v,w)$ is:

$$r(e) := c(e) + \phi(v) - \phi(w)$$

Lemma 2: Let p and q be paths connecting the same endpoints in G. Then for every potential ϕ: $r(p)<r(q)$ if and only if $c(p)<c(q)$.

Johnson´s Method

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Proof: Let $p=(v_1,\ldots,v_k)$ be an arbitrary path and $e_i=(v_i,v_{i+1})$ for all i. It holds:

$$\begin{aligned} r(p) &= \sum_i r(e_i) \\ &= \sum_i (\phi(v_i) + c(e_i) - \phi(v_{i+1})) \\ &= \phi(v_1) + c(p) - \phi(v_k) \end{aligned}$$

Johnson´s Method

Lemma 3: Suppose that G has no negative cycles and that all nodes can be reached from s. Let $\phi(v)=\delta(s,v)$ for all $v \in V$. With this ϕ, $r(e) \geq 0$ for all e.

Proof:

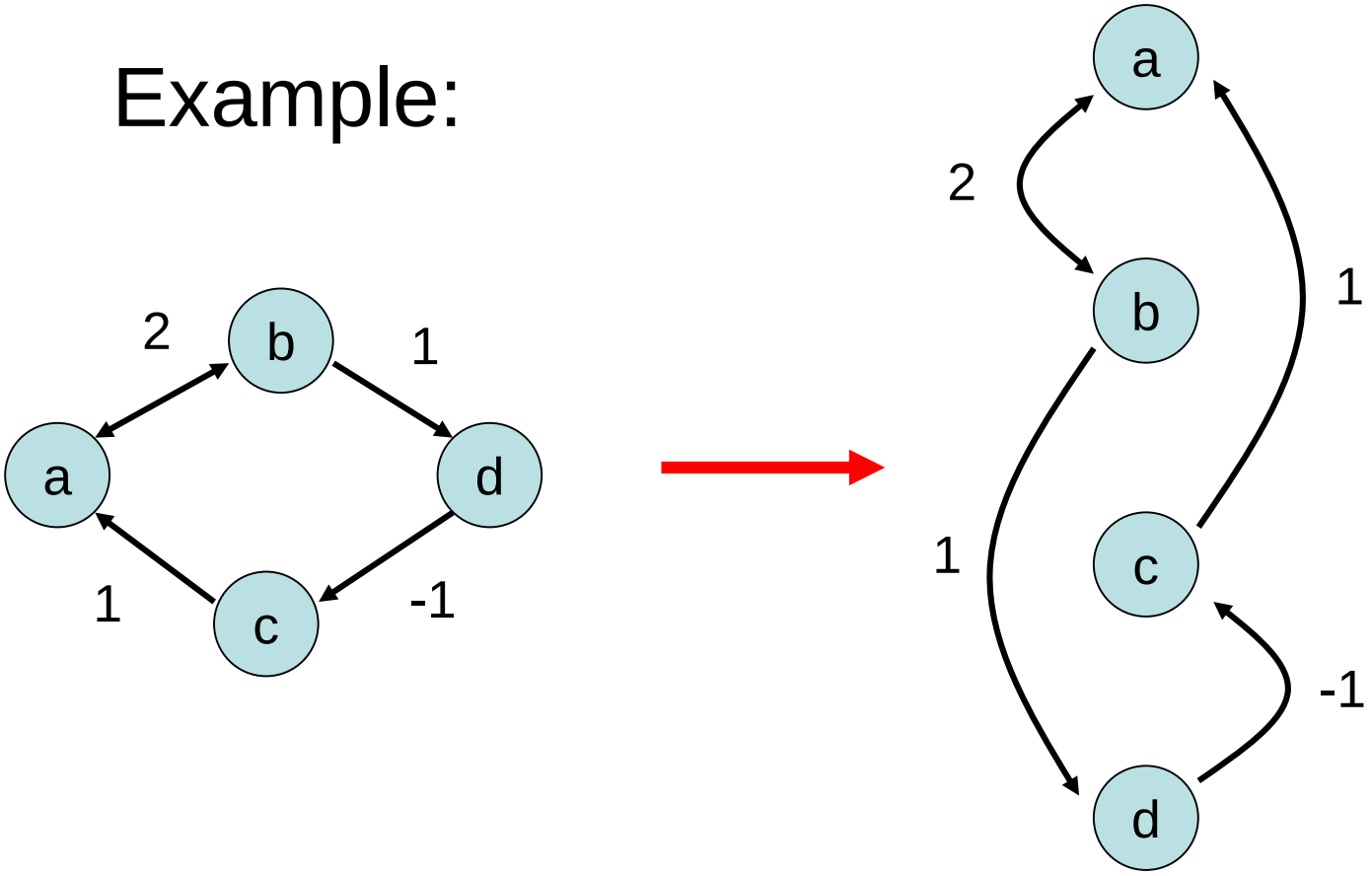
- According to our assumption, $\delta(s,v) \in \mathbb{R}$ for all v
- We know: for every edge $e=(v,w)$, $\delta(s,v)+c(e) \geq \delta(s,w)$ (otherwise, we have a contradiction to the definition of δ!)
- Therefore, $r(e) = \delta(s,v) + c(e) - \delta(s,w) \geq 0$

Johnson´s Method

1. Create new node s and new edges (s,v) for all v in G with $c(s,v)=0$ (all nodes reachable!)
2. Compute $\delta(s,v)$ using Bellman-Ford and set $\phi(v):=\delta(s,v)$ for all v
3. Compute the reduced costs $r(e)$
4. Compute for all nodes v the distances $\bar{\delta}(v,w)$ using Dijkstra with the reduced costs on graph G without node s
5. Compute the correct distances $\delta(v,w)$ via $\delta(v,w):=\bar{\delta}(v,w)+\phi(w)-\phi(v)$

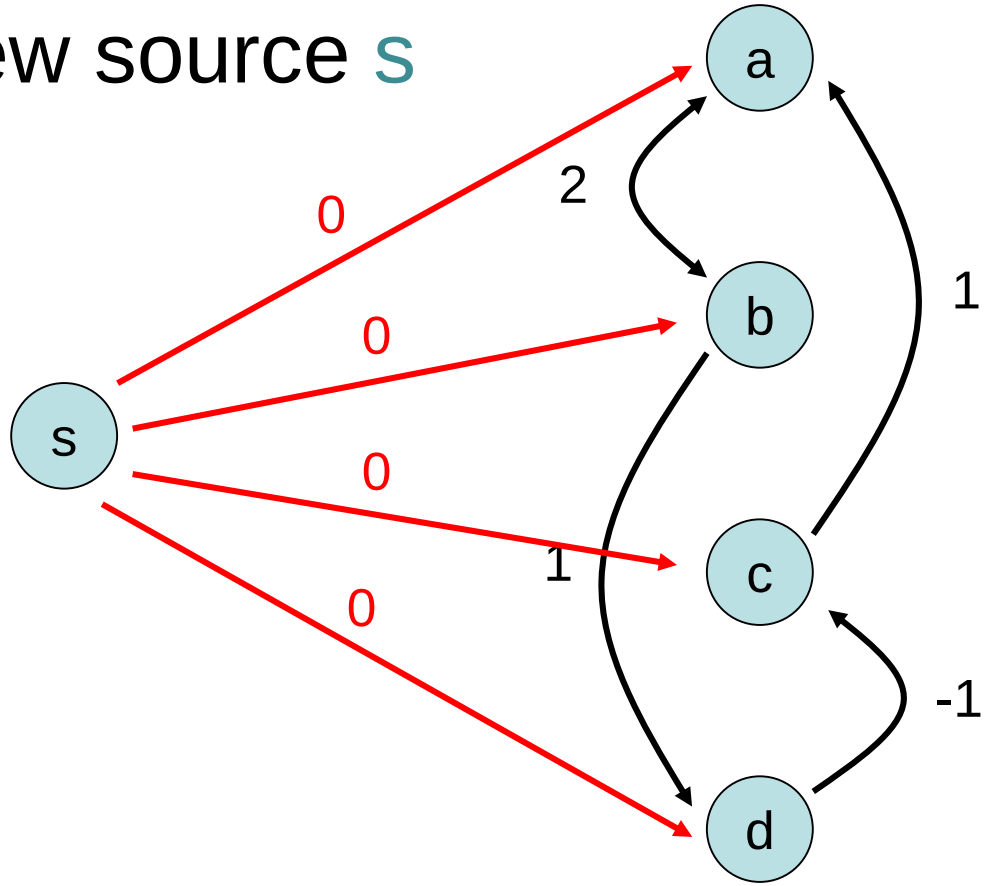
Johnson´s Method

Example:



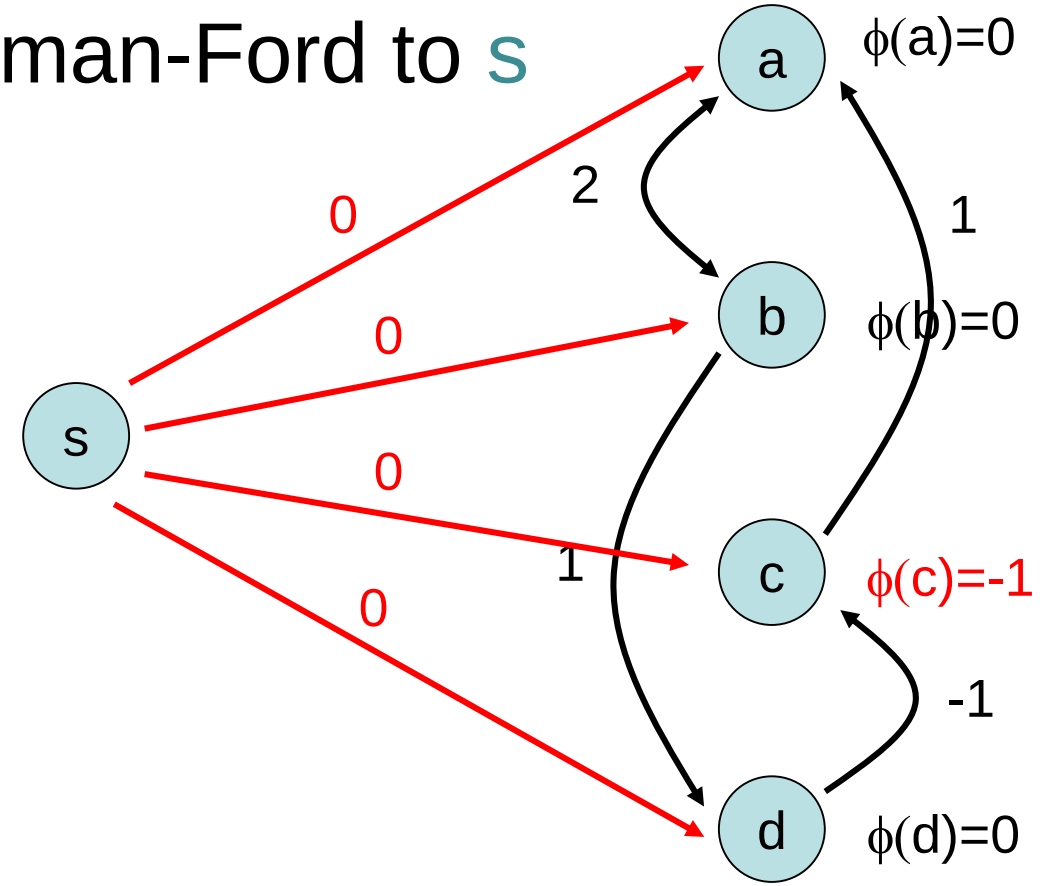
Johnson´s Method

Step 1: create new source s



Johnson´s Method

Step 2: apply Bellman-Ford to s

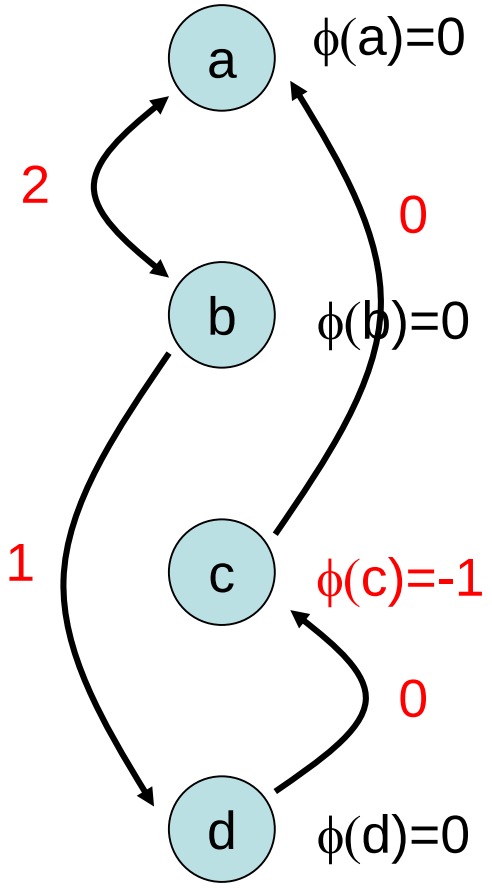


Johnson´s Method

Step 3: compute r(e)-values

The reduced cost of e=(v,w) is:

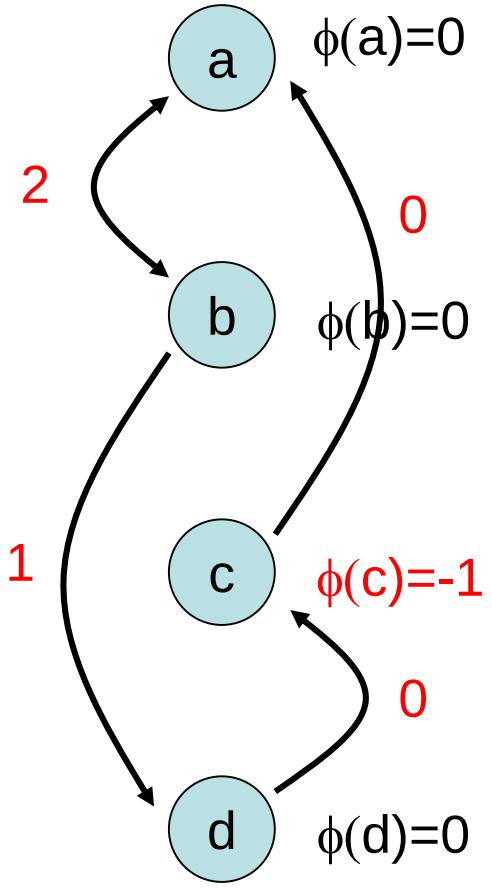
$$r(e) := \phi(v) + c(e) - \phi(w)$$



Johnson´s Method

Step 4: compute all distances $\bar{\delta}(v,w)$ via Dijkstra

$\bar{\delta}$	a	b	c	d
a	0	2	3	3
b	1	0	1	1
c	0	2	0	3
d	0	2	0	0

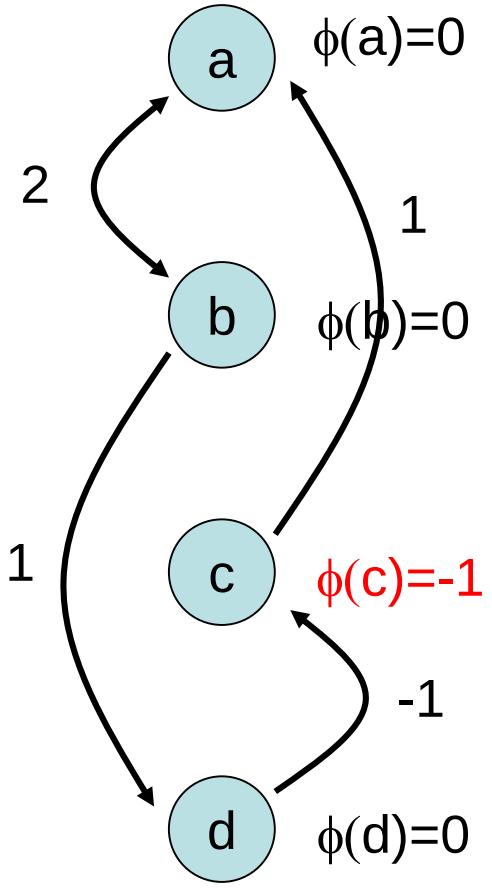


Johnson´s Method

Step 5: compute correct distances via the formula

$\delta(v,w)=\bar{\delta}(v,w)+\phi(w)-\phi(v)$

δ	a	b	c	d
a	0	2	2	3
b	1	0	0	1
c	1	3	0	4
d	0	2	-1	0



All Pairs Shortest Paths

Runtime of Johnson´s Method:

$$O(T_{\text{Bellman-Ford}}(n,m) + n \cdot T_{\text{Dijkstra}}(n,m))$$
$$= O(n \cdot m + n(n \log n + m))$$
$$= O(n \cdot m + n^2 \log n)$$

when using Fibonacci heaps.

- Problem with the runtime bound: m can be quite large in the worst case (up to $\sim n^2$)
- Nevertheless, best known bound: $O(n \cdot m)$