Advanced Distributed Algorithms and Data Structures

Chapter 2: Graph Theory

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Overview

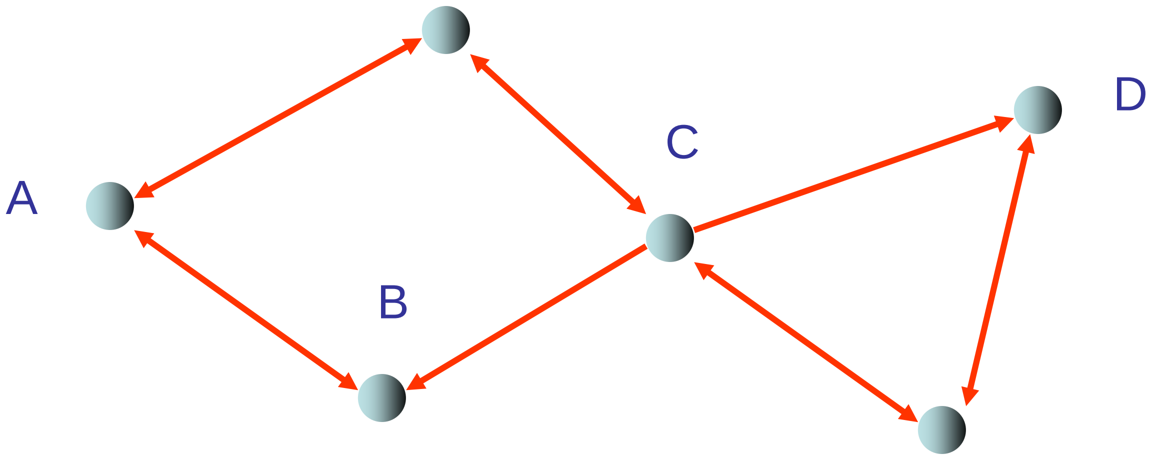
- Graph theory
- Classical graph families
- Fundamental graph parameters

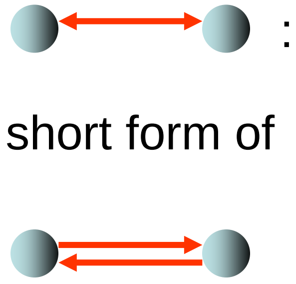
Graph Theory

Definition 2.1: A graph $G=(V,E)$ consists of a node set V and an edge set E.

- G undirected: $E \subseteq \{ \{v,w\} \mid v,w \in V\}$

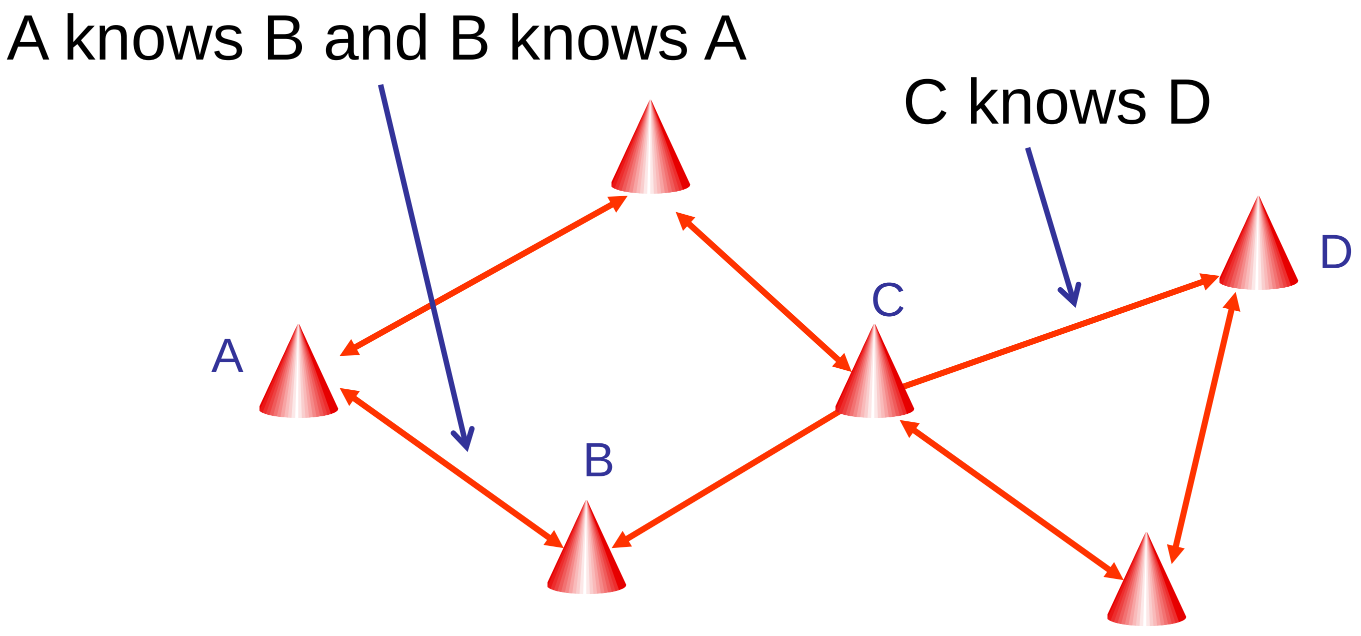
- G directed: $E \subseteq \{ (v,w) \mid v,w \in V\}$





Graph Theory

In our case: graph represents knowledge or connections between processes



Graph Theory

Definition 2.2: Let $G=(V,E)$ be a graph.

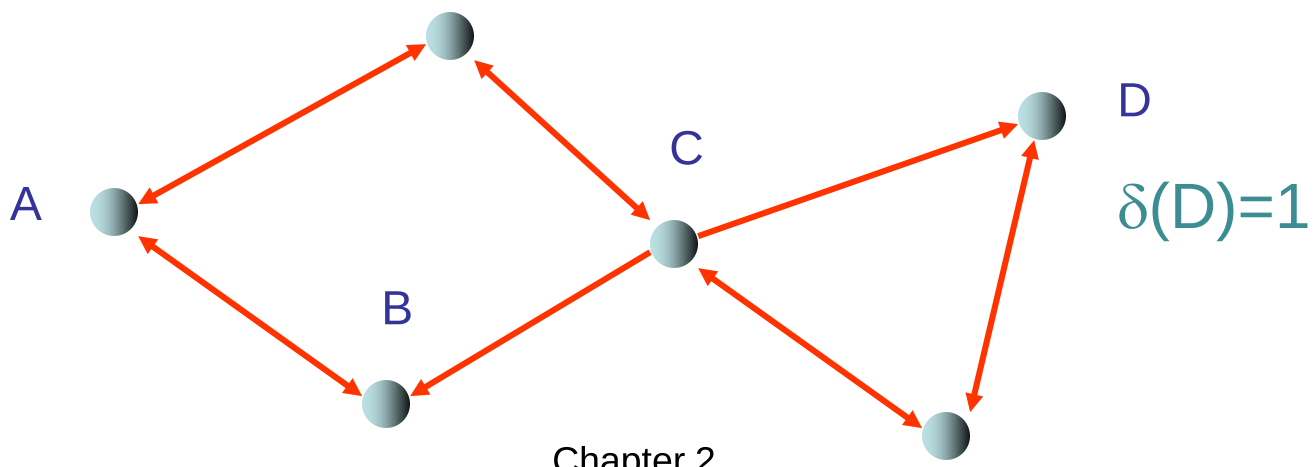
- G undirected: degree of $v \in V$:

$$\delta(v)=|\{ w \in V \mid \{v,w\} \in E\}|$$

- G directed: degree of $v \in V$:

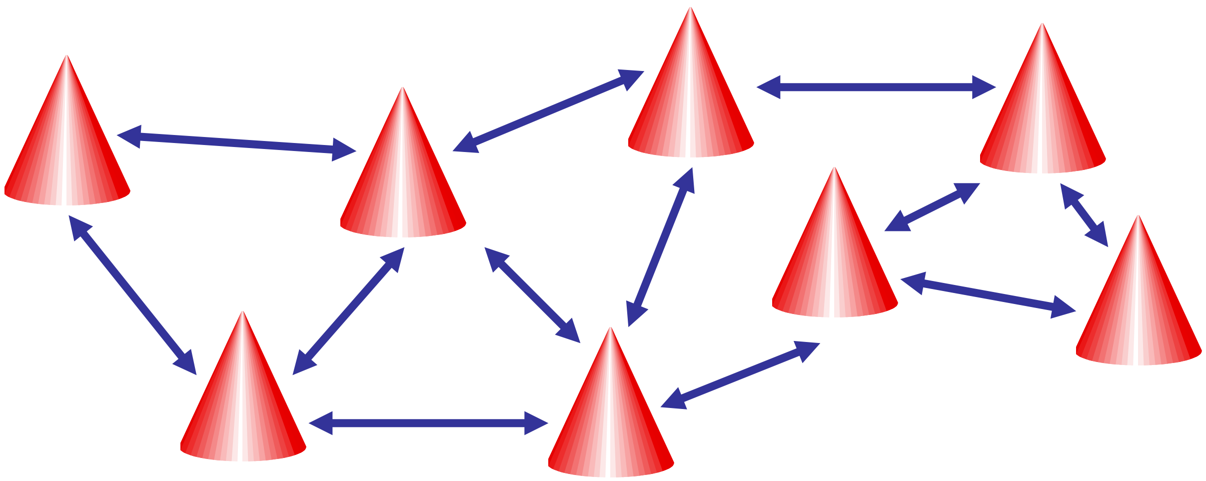
$$\delta(v)=|\{ w \in V \mid (v,w) \in E\}|$$

Degree of G: $\Delta = \max_{v \in V} \delta(v)$



Graph Theory

Degree: corresponds to update costs for processes if the set of processes changes.

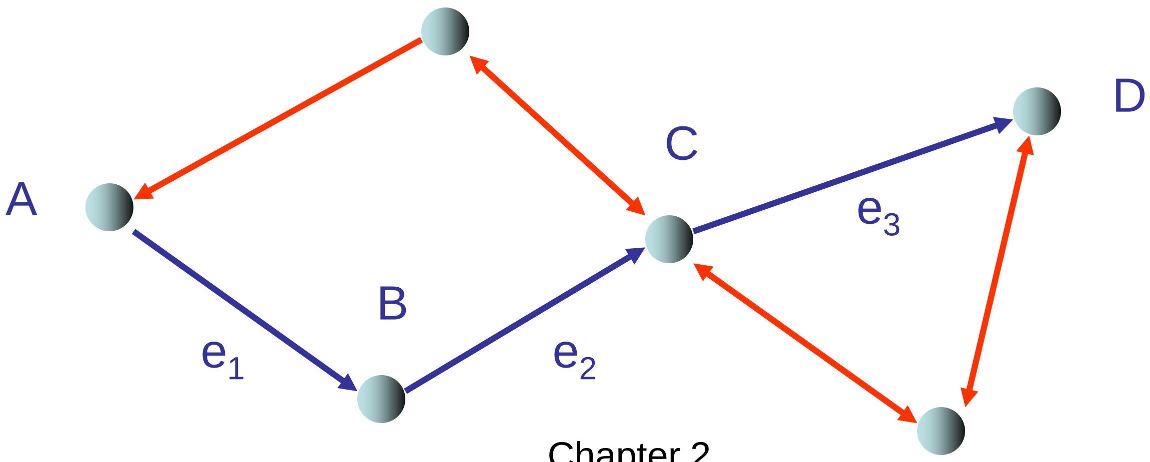


Degree should not be too high.

Graph Theory

Definition 2.3: Let $G=(V,E)$ be a graph. An edge sequence $p=(e_1,e_2,\dots,e_k)$ in G is called a path if there is a node sequence $(v_0,\dots,v_k)$ with

- G undirected: $e_i=\{v_{i-1},v_i\}$ for all $i\in\{1,\dots,k\}$
- G directed: $e_i=(v_{i-1},v_i)$ for all $i\in\{1,\dots,k\}$



Graph Theory

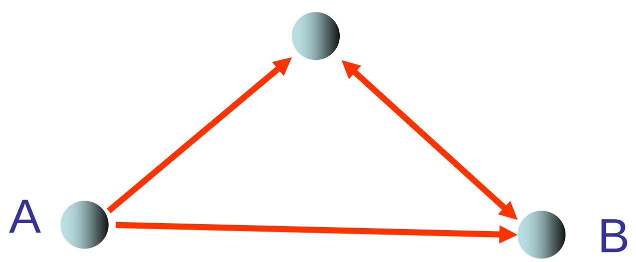
Definition 2.4: A graph $G=(V,E)$ is called

- connected if G is undirected and for all node pairs $v,w \in V$ there is a path from v to w in G.
- weakly connected if G is directed and for all node pairs $v,w \in V$ there is a path from v to w in the undirected version of G.
- strongly connected if G is directed and for all node pairs $v,w \in V$ there is a (directed) path from v to w in G.

Graph Theory

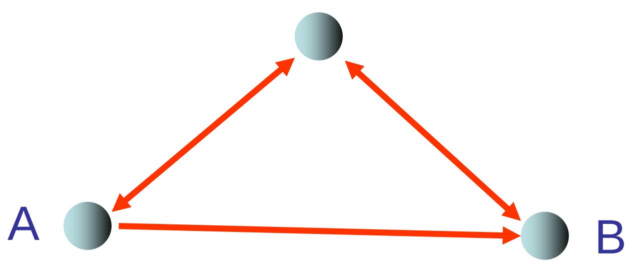
Examples:

(1) Graph is only weakly connected



no directed path
from B to A

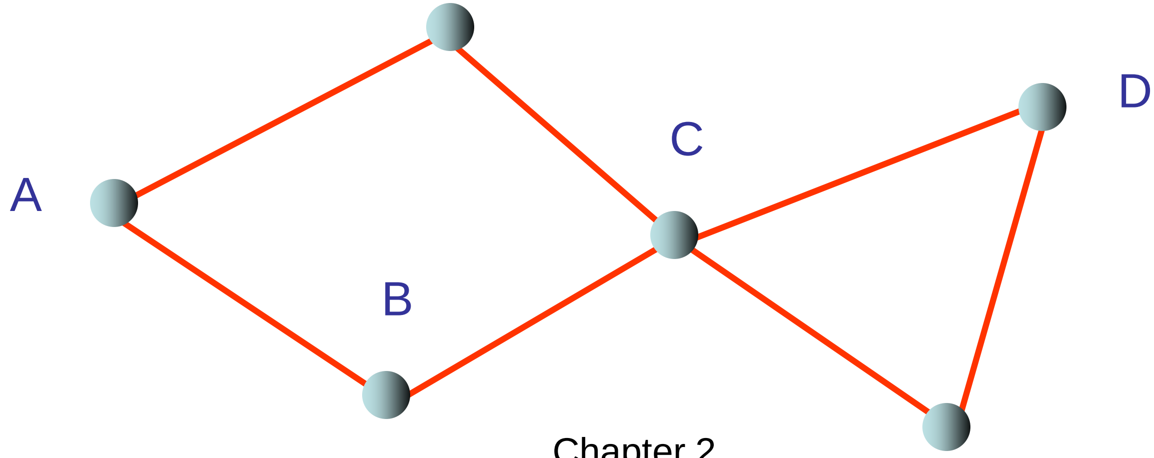
(2) Graph is strongly connected



Graph Theory

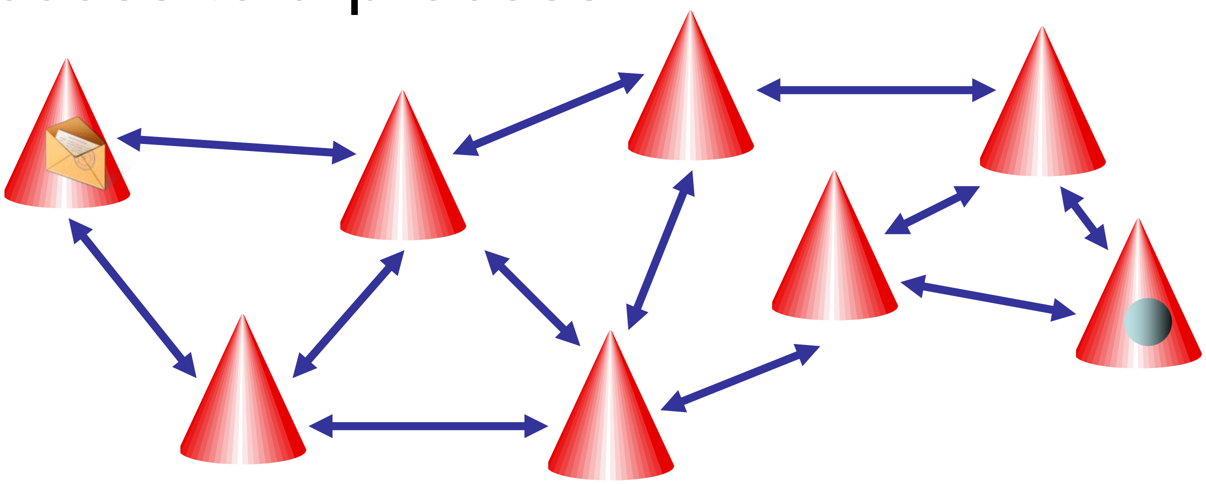
Definition 2.5: Let $G=(V,E)$ be a graph and $p=(e_1,e_2,\ldots,e_k)$ be a path from v to w in G.

- Length of p: $|p|=k$
- Distance of w from v: $d(v,w)$ = min. path length from v to w ($d(v,w) = \infty$ if there is no path from v to w)
- Diameter of G: $D(G)=\max_{v,w\in V} d(v,w)$



Graph Theory

Diameter: lower bound for worst-case time (measured in number of communication rounds) for access to a process

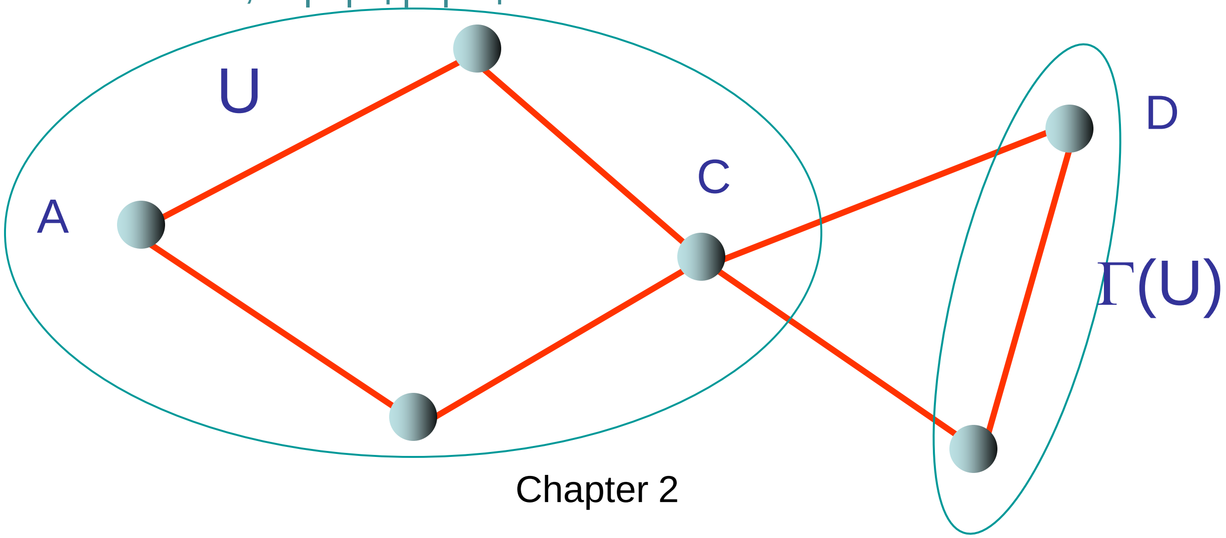


Diameter should not be too high.

Graph Theory

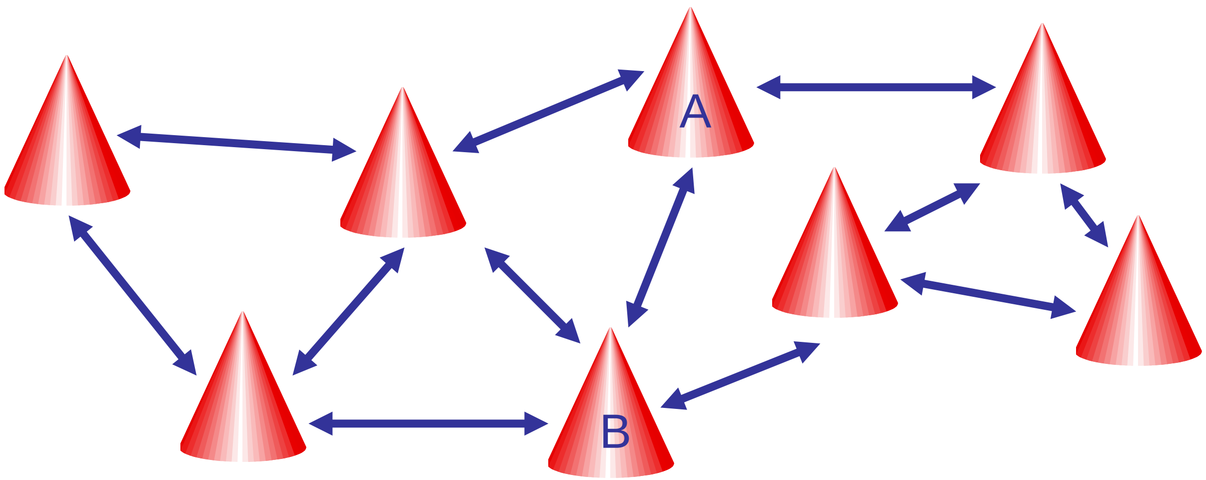
Definition 2.6: Let $G=(V,E)$ be a graph.

- $\Gamma(U)$: neighbor set of a node set $U \subseteq V$, i.e.,
  $\Gamma(U) = \{ w \in V \setminus U \mid$ there is a $v \in U$ with $\{v,w\} \in E$ (resp. $(v,w) \in E$ in the directed case) $\}$
- $\alpha(U) = |\Gamma(U)| / |U|$: expansion of U
- $\alpha(G) = \min_{U \subseteq V, 1 \leq |U| \leq \lceil |V|/2 \rceil} \alpha(U)$: expansion of G



Graph Theory

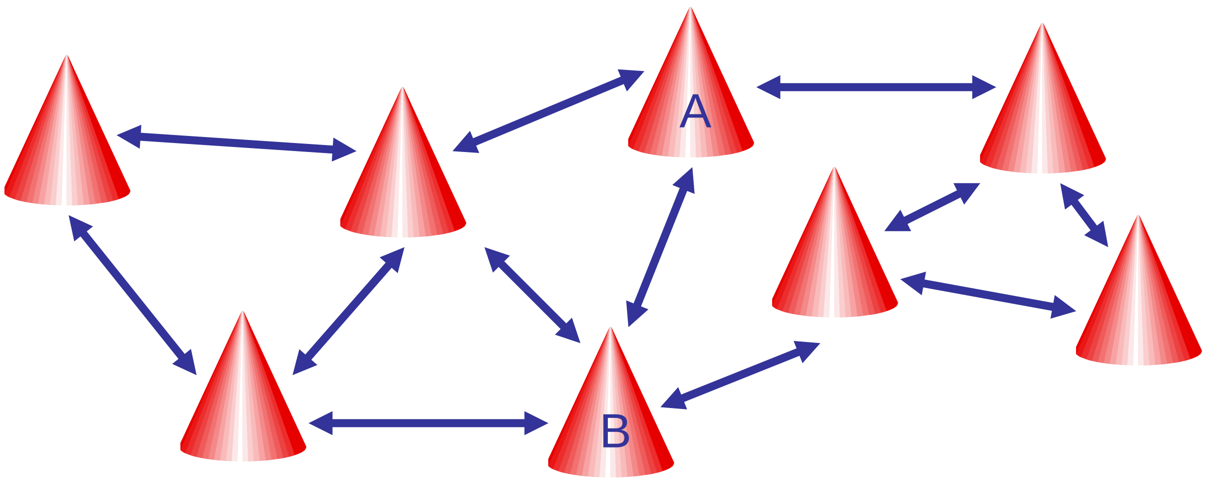
Expansion: k failures $\Rightarrow$ at most $k/\alpha(G)$ nodes get disconnected from rest of the graph



Proof: Let U be set of all non-failing nodes that get disconnected due to failed nodes. Then all nodes in $\Gamma(U)$ failed, i.e., $|\Gamma(U)| \leq k$. Moreover, $\alpha(U) \geq \alpha(G)$ and $\alpha(U) = |\Gamma(U)|/|U|$, so $|U| \leq k/\alpha(G)$.

Graph Theory

Expansion: k failures $\Rightarrow$ at most $k/\alpha(G)$ nodes get disconnected from rest of the graph

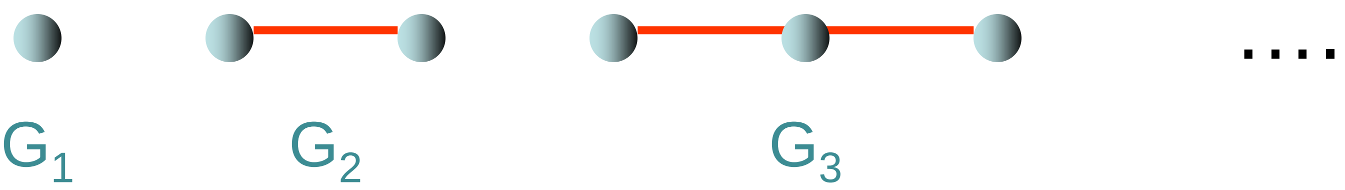


Expansion should be as high as possible

Graph Theory

In the following we consider classical families of graphs $\mathcal{G}=\{G_1, G_2, \ldots\}$.

Example: Family of linear lists

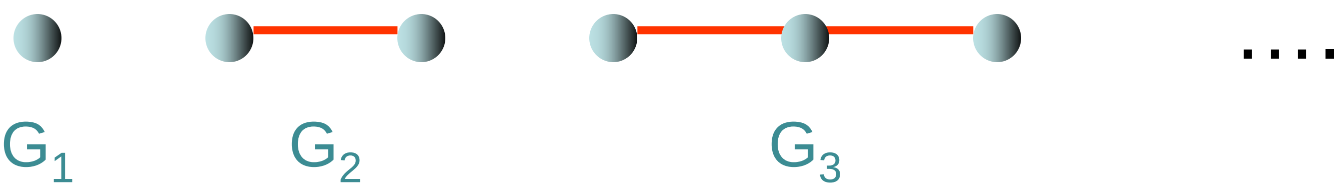


We say: graph G from a family $\mathcal{G}$ has constant degree if the degree of all graphs in $\mathcal{G}$ is upper bounded by a constant.

Graph Theory

In the following we consider classical families of graphs $\mathcal{G}=\{G_1, G_2, \ldots\}$.

Example: Family of linear lists

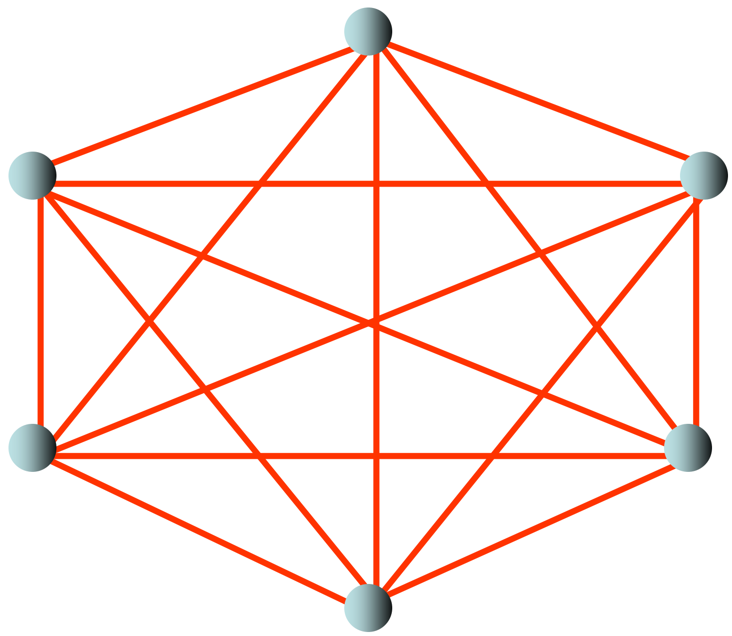


For a graph G from $\mathcal{G}$ we use

- n: number of nodes (resp. size) of G
- m: number of edges of G

Classical Graph Families

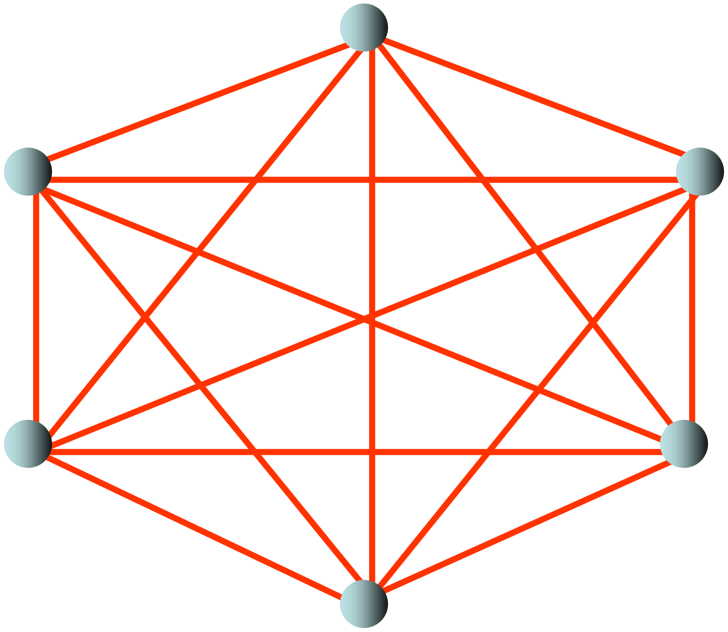
Complete graph / clique: every node is connected to every other node



Advantage: low diameter, high expansion

Classical Graph Families

Complete graph / clique: every node is connected to every other node



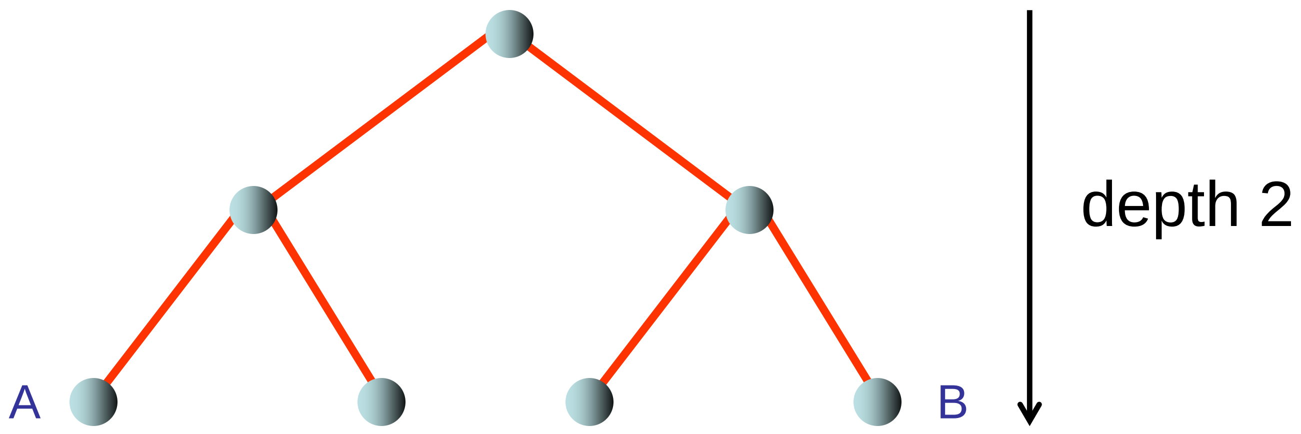
Problem: high degree! ($\delta(v)=n-1$ for all v)

Linear List

- Degree 2 (minimal for connectivity), BUT
- Diameter is bad ($D(List)=n-1$)
- Expansion is bad ($\alpha(List)\approx 2/n$)

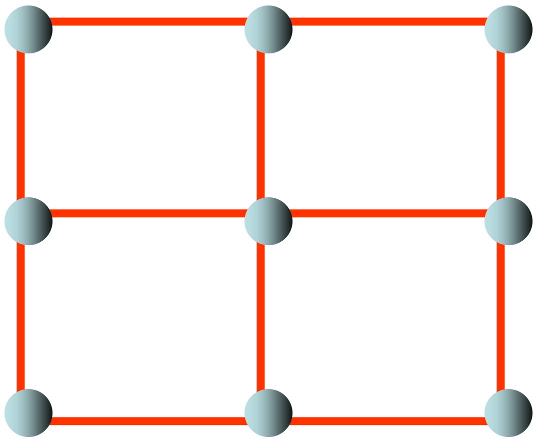
How to obtain a small degree and diameter?

Complete Binary Tree



- $n=2^{k+1}-1$ nodes when depth is $k \in \mathbb{N}_0$
- degree is 3
- Diameter is $2k \approx 2 \log_2 n$, BUT
- Expansion is bad ($\alpha(\text{Baum}) \approx 2/n$)

2-dimensional Grid

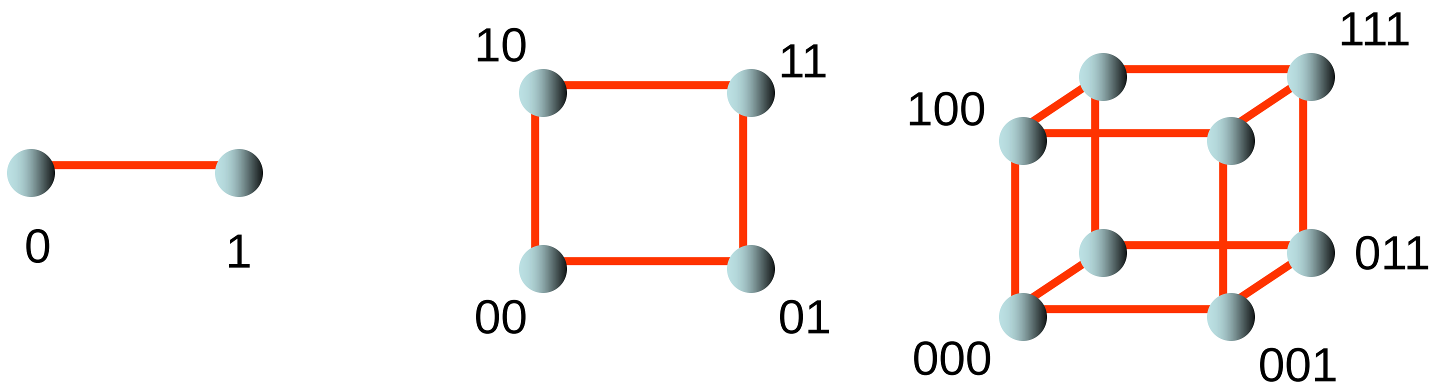


- $n = k^2$ nodes when there are k nodes along each side, maximal degree 4
- Diameter is $2(k-1) \approx 2\sqrt{n}$
- Expansion is $\approx 2/\sqrt{n}$
- Not bad, but can we do better?

d-dimensional Hypercube

- Nodes: $(x_1,\dots,x_d) \in \{0,1\}^d$
- Edges: $\forall i$: $(x_1,\dots,x_d) \rightarrow (x_1,..,1\text{-}x_i,..,x_d)$

Bit i flipped

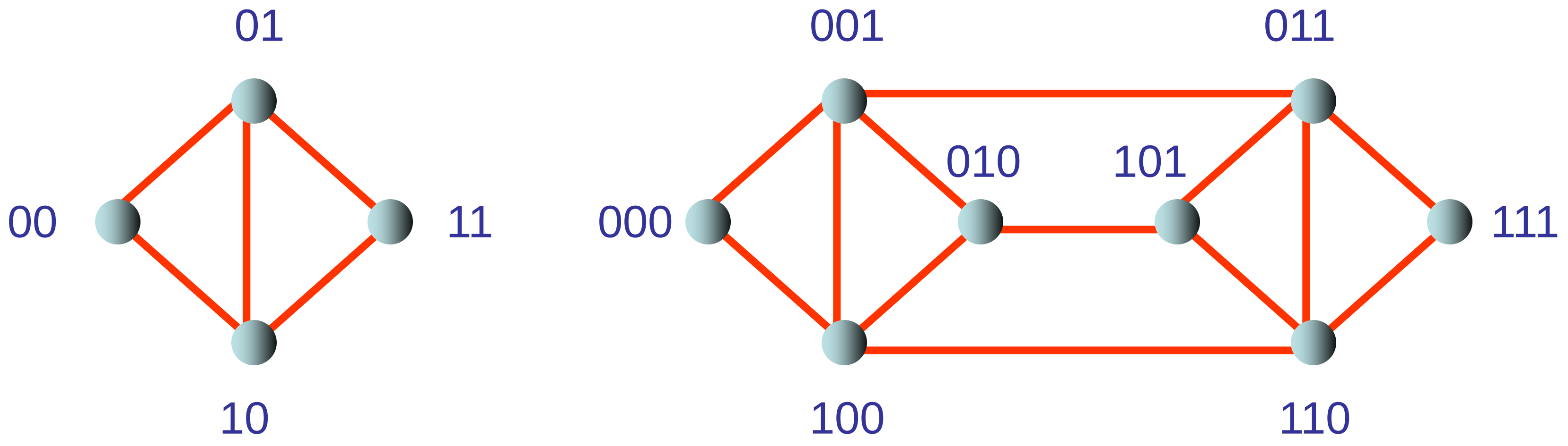


d=1 d=2 d=3

Degree d, diameter d, expansion $\approx 1/\sqrt{d}$

d-dimensional de Bruijn Graph

- Nodes: $(x_1,\ldots,x_d) \in \{0,1\}^d$
- Edges: $(x_1,\ldots,x_d) \rightarrow (0,x_1,x_2,\ldots,x_{d-1})$
  $(1,x_1,x_2,\ldots,x_{d-1})$



Diameter

Theorem 2.7: Every graph of maximal degree $\delta>2$ and size n has a diameter of at least $(\log n)/(\log(\delta-1))-1$.

Proof: exercise

Theorem 2.8: For all even $\delta>2$ there is a family of graphs of maximal degree δ and size n with diameter at most $(\log n) / (\log \delta -1)$.

Proof: exercise

Expansion

Theorem 2.9: For every graph G it holds that $\alpha(G) \in [0,1]$.

Proof: see the definition of the expansion $\alpha(G)$.

Theorem 2.10: There is a family of graphs with constant degree and constant expansion.

Example: Gabber-Galil Graph

- Node set: $(x,y) \in \{0,\ldots,k-1\}^2$
- $(x,y) \rightarrow (x,x+y),(x,x+y+1), (x+y,y), (x+y+1,y) \pmod{k}$

References

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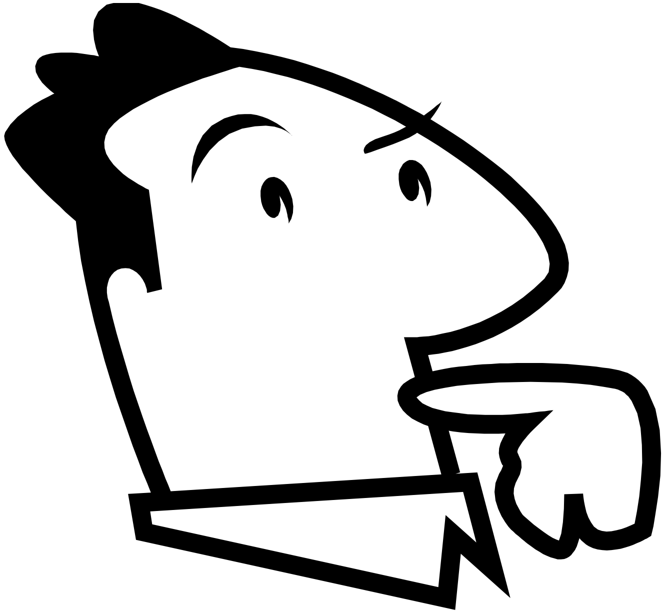

Questions?