Prof. Dr. Christian Scheideler
Universität Paderborn

Paderborn, October 22, 2019

Advanced Algorithms

WS 2019

Homework Assignment 3

Problem 8:
Prove Lemma 2.9.

Problem 9:
In the well-known skip list data structure, a set of n elements is arranged in a set of sorted lists $L_0, L_1, L_2, \ldots$, where L_0 is the sorted list containing all elements. In addition to that, every element e_i chooses a random bit vector x_i, and element e_i participates in list L_k if and only if the first k bits in x_i are 1.

(a) What is the expected index of the highest list that element e_i participates in?

(b) Show that with high probability (i.e., a probability of at least $1 - 1/n$) the highest index k of a list L_k that contains elements is $O(\log n)$.

(c) Propose a search operation that would reach any element in (expected) $O(\log n)$ time when starting with the first element in L_0. (A formal analysis of the runtime is not needed.)

Problem 10:
Consider the situation that we have n processes, where process i initially stores some number $x_i \in \mathbb{N}$. The goal for the n processes is to compute the minimum of these numbers. In order to do so, they execute the following algorithm in synchronized rounds:
In each round, each process i contacts a process j uniformly and independently at random and requests the number x_j currently stored in j. It then sets $x_i := \min\{x_i, x_j\}$.
Our goal is to prove that at most $O(\log n)$ rounds are needed till all processes know the minimum. For that we separate the analysis into three stages, where the following needs to be shown:

(a) With probability at least $1 - 1/n$, after $O(\log n)$ rounds at least $c \log n$ many processes will know the minimum (where c is a sufficiently large constant so that (b) works).

(b) If at the beginning of a round, k many processes know the minimum, where $c \log n \leq k \leq n/2$, then with probability at least $1 - 1/n$ at least $(5/4)k$ processes will know the minimum at the end of the round.

(c) If at the beginning of a round, k many processes do *not* know the minimum yet, where $k \leq n/2$, then with probability at least $1 - 1/n$ at most $(3/4)k$ processes will not know the minimum yet at the end of the round.

Hint: properly define binary random variables and use the Chernoff bounds in each stage.