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# Advanced Algorithms

WS 2019

## Homework Assignment 11

**Problem 28:**
Let $S$ be a set of size $n$ and $\phi : 2^S \to \mathbb{R}$ be a function that maps any set $R \subseteq S$ to some value $\phi(R)$. Define

$$\begin{aligned} V(R) &:= \{s \in S \setminus R \mid \phi(R \cup \{s\}) \neq \phi(R)\} \\ X(R) &:= \{s \in R \mid \phi(R \setminus \{s\}) \neq \phi(R)\} \end{aligned}$$

$V(R)$ is the set of *violators* of $R$ and $X(R)$ is the set of *extreme elements* in $R$. Certainly,

$$s \text{ violates } R \Leftrightarrow s \text{ is extreme in } R \cup \{s\}$$

For a random subset $R \subseteq S$ of size $r$ we define the random variables $V_r = |V(R)|$ and $X_r = |X(R)|$. Use your insights from Chapter 7 to show:
For all $0 \leq r < n$,

$$\frac{\mathbb{E}[V_r]}{n-r} = \frac{\mathbb{E}[X_{r+1}]}{r+1}$$

**Problem 29:**
A *violator space* is a pair $(H, V)$ where $H$ is a finite set and $V$ is mapping $2^H \to 2^H$ such that the following two conditions are fulfilled:

- *Consistency*: For all $G \subseteq H$, $G \cap V(G) = \emptyset$.
- *Locality*: For all $F \subseteq G \subseteq H$ where $G \cap V(F) = \emptyset$, $V(G) = V(F)$.

Show that any violator space $(H, V)$ satisfies monotonicity defined as follows:

- *Monotonicity*: $V(F) = V(G)$ implies $V(E) = V(F) = V(G)$ for all sets $F \subseteq E \subseteq G \subseteq H$.

In fact, violator spaces are the most general form of abstract optimization problems to which Clarkon's algorithms can be applied.

**Problem 30:**
Consider any integer linear program $P$ with objective function $f(x) = c^T \cdot x$ and constraints $Ax \leq b$ that has a finite number of solutions. Let $\#P$ be the problem of counting the number of feasible solutions for $P$, i.e., the number of vectors $x \in \mathbb{Z}^n$ that satisfy $Ax \leq b$. Show that if $\#P$ can be solved in polynomial time then the optimal solution of $P$ can be found in polynomial time.
Hint: You may assume that any $x$ satisfying $Ax \leq b$ only consists of values $x_i$ where $|x_i|$ is at most exponentially large in the input size.

**Problem 31:**
Prove Theorem 8.3.